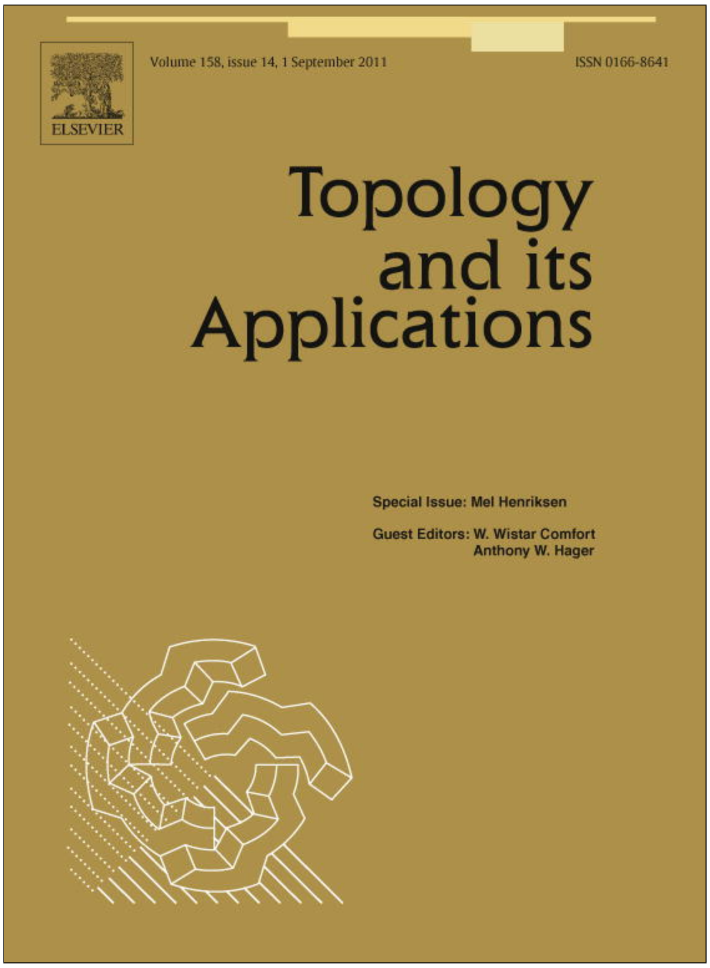
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Remembering Mel Henriksen and (some of) his theorems

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ABSTRACT

The author selects theorems from three papers co-authored by Mel Henriksen, proves some of those, and offers some consequences and commentary. Also included are some comments, mathematical and social, on Mel Henriksen as a colleague, a co-author, and a forceful presence in the wider political and mathematical community.

Mel Henriksen cut a wide swath, traveling widely in the pursuit and promotion of mathematical truth. His interests—mathematical, social and political—were similarly widespread. He brought the same honesty to his mathematical research that he brought to real-world issues. Mel hated anything that smelled of double-talk. Devious or dishonest behavior on the part of a student, or a colleague, or an administrator, or a stuffy editor, or on the part of the hierarchy of the American Mathematical Society, was likely to be rewarded by a penetrating, well-reasoned, scathing, dressing-down, or by a hard-nosed Letter to the Editor. You couldn't spend time with Mel without profiting, if that's the right word, from a bit of constructive commentary about your thought processes, your ethical values, your lack of attention to your fellow man. But the key word there is *constructive*. To my knowledge there was not a mean-spirited bone in Mel's body. He was gentle and encouraging, and when he sensed that you were doing your best he did not ridicule you.

Especially in the last decade or two of his life and at the expense of much personal time, Mel worked creatively to encourage and foster mathematical activity in corners of the earth where facilities and authoritative research information was minimal or unavailable. He was a generous and caring man, always honorably motivated. I admired Mel, I always enjoyed spending time with him, and I miss him.

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[1] This article derives from an address delivered March 27, 2010 to the Mel Henriksen Memorial Conference held that day at Harvey Mudd College in Claremont California. I am pleased to thank Norman Noble and Thomas J. Peters, who offered several helpful comments on an early version of this manuscript.

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Scientifically, Mel returned inexorably and repeatedly to his principal love, the study of rings of continuous functions with emphasis on ideals with special properties and their quotients. In this article, however, deferring to other contributors whose career and research are more closely linked to Mel's than mine, I will follow Mel slightly afield into three regions of set-theoretic inquiry he visited only briefly. My choice of these three papers is idiosyncratic. There was much to choose from.

1. The paper [15]

When a careful history of the development of perfect functions is written, surely the names of Vaĭstein [22,23], Leray [18], Whyburn [24], and Frolík [10] will figure prominently. P.S. Alexandroff [2] (§5, especially footnote 1 on page 55) gives a helpful historical perspective. For obvious reasons I will focus here on the fundamental results given by Henriksen and Isbell [15] in 1957.

To simplify the discussion, and because it will be convenient to use properties of the Stone–Čech compactification, I restrict attention in this section (except in a brief discussion preceding Corollary 1.4) to Tychonoff spaces. I will use not the terminology introduced in [15] but that which has become generally accepted by later workers.

Definition 1.1. A surjective function $f : X \twoheadrightarrow Y$ is *perfect* if

(a) f is continuous (we write $f \in C(X, Y)$);
(b) A closed in $X \Rightarrow f[A]$ closed in Y; and
(c) $y \in Y \Rightarrow f^{-1}(\{y\})$ is compact.

The utility of this concept becomes evident upon reading the following theorem.

Theorem 1.2. *([15]) Let X and Y be spaces and $f : X \twoheadrightarrow Y$ a continuous surjection. Then f is perfect if and only if its Stone extension $\overline{f} : \beta X \twoheadrightarrow \beta Y$ satisfies $\overline{f}[\beta X \backslash X] = \beta Y \backslash Y$.*

Besides in the paper being lauded here, the proof of Theorem 1.2 has been recorded frequently in the literature. See for example [9, (3.7.16)] and [7, (9.2)]. The following theorem, of which most parts appear already in [15], offers an incomplete sample of the many consequences of Theorem 1.2.

Theorem 1.3. *Let $f : X \twoheadrightarrow Y$ be perfect and let $\mathbb{P}$ be one of the following properties. If Y has $\mathbb{P}$, then X has $\mathbb{P}$:*

(a) *compact*;
(b) *Lindelof*;
(c) *realcompact*;
(d) *paracompact*;
(e) *topologically complete*;
(f) *σ-compact*;
(g) *Čech-complete (i.e., is a G_δ-set in its Stone–Čech compactification).*

Proof. (a)–(f) graph($\overline{f}$) is closed in $\beta X \times \beta Y$ and is homeomorphic to βX, so $\overline{f}^{-1}(Y)$ is homeomorphic to a closed subspace of $\beta X \times Y$. And, $\overline{f}^{-1}(Y) = X$. It is enough to recall then that (1) the product of a compact space with a space with $\mathbb{P}$ again has $\mathbb{P}$ and (2) within the class of Tychonoff spaces, property $\mathbb{P}$ is inherited by closed subspaces.

(g) Let $Y = \bigcap_n U_n$, with U_n open in βY. Then $X = \bigcap_n \overline{f}^{-1}(U_n)$, a G_δ in βX. □

For X a space, typically but not necessarily Tychonoff, and for $f \in \mathbb{R}^X$, we set $Z(f) := \{x \in X\colon\ f(x) = 0\}$ and $\mathrm{coz}(f) := X \backslash Z(f)$; and we write $\mathcal{Z} = \mathcal{Z}(X) := \{Z(f)\colon\ f \in C(X, \mathbb{R})\}$. The sets $Z(f)$ with $f \in C(X, \mathbb{R})$ are the *zero-sets* of X, and the sets $X \backslash Z(f)$ are *cozero-sets* of X. The following pleasing corollary to Theorems 1.2 and 1.3, due to Frolík [10], itself has a number of useful consequences; see for example [10] and [7]. The direction $\Leftarrow$ in its proof is routine while $\Rightarrow$, although more sophisticated, practically "writes itself" when the appropriate tools are assembled. We content ourselves with an outline only, referring the reader to [7, (9.4)] for full details.

Corollary 1.4. *([10]) A Tychonoff space X is paracompact and Čech-complete if and only if there are a complete metric space M and a perfect map $f : X \twoheadrightarrow M$.*

Proof. ($\Leftarrow$) A complete metric space is a G_δ-set in every space containing it densely; in particular, then, M is a G_δ-set in βM. Now, use (d) and (g) of Theorem 1.3.

(⇒) Let $\beta X\backslash X = \bigcup_n K_n$, with each K_n compact. For each n, there is an open cover $\mathcal{U}_n$ of X such that $(\mathrm{cl}_{\beta X} U) \cap K_n = \emptyset$ for each $U \in \mathcal{U}_n$. According to a theorem of Michael [19], each $\mathcal{U}_n$ has a locally finite (in X) cozero refinement $\mathcal{V}_n$; for $V \in \mathcal{V}_n$ write $V = \mathrm{coz}(f_V)$ with $f_V \in C(X, [0,1])$. Now define:

1. $d_n(x, x') := \min\{1, \sum_{V \in \mathcal{V}_n} |f_V(x) - f_V(x')|\}$;
2. $d := \sum_n d_n/2^n$;
3. $\bar{x} := \{x' \in X\colon\ d(x, x') = 0\}$;
4. $M := \{\bar{x}\colon\ x \in X\}$;
5. $\rho(\bar{x}, \overline{x'}) := d(x, x')$; and
6. $f : X \twoheadrightarrow M$ by $f(x) = \bar{x}$.

One then easily shows that (M, ρ) is a metric space and that f is continuous with $\overline{f}[\beta X\backslash X] = \beta M\backslash M$, so f is perfect by Theorem 1.2. As a G_δ-set in βM, M is completely metrizable. □

2. The paper [16]

In preparation for the material below concerning Baire sets I first recall some set-theoretic terminology and constructions.

Definition 2.1. Let Y be a set. A family $\mathcal{A} \subseteq \mathcal{P}(Y)$ is a *σ-lattice* if $\mathcal{S} \subseteq \mathcal{A}$, $|\mathcal{S}| \leqslant \omega \Rightarrow \bigcup \mathcal{S} \in \mathcal{A}$ and $\bigcap \mathcal{S} \in \mathcal{A}$; a *σ-algebra* $\mathcal{A} \subseteq \mathcal{P}(Y)$ is a σ-lattice in which $A \in \mathcal{A} \Rightarrow Y\backslash A \in \mathcal{A}$.

Remarks 2.2. (a) Clearly when $\mathcal{B} \subseteq \mathcal{P}(Y)$, there is a least σ-lattice $\sigma\text{-}lat(\mathcal{B}) \subseteq \mathcal{P}(Y)$ containing $\mathcal{B}$. Similarly there is a least σ-algebra $\sigma\text{-}alg(\mathcal{B}) \subseteq \mathcal{P}(Y)$ containing $\mathcal{B}$.

(b) As the definition makes clear, when a family $\mathcal{B}$ of sets is given the family $\sigma\text{-}lat(\mathcal{B})$ is independent of the choice of Y (with $\mathcal{B} \subseteq \mathcal{P}(Y)$). Not so for $\sigma\text{-}alg(\mathcal{B})$.

(c) If $A \in \mathcal{B} \Rightarrow Y\backslash A \in \mathcal{B}$ then $\sigma\text{-}lat(\mathcal{B})$ is a σ-algebra, so $\sigma\text{-}lat(\mathcal{B}) = \sigma\text{-}alg(\mathcal{B})$.

The following definitions are familiar.

Definition 2.3. Let $Y = (Y, \mathcal{T})$ be a topological space. Then

$$\mathrm{Borel}(Y) := \sigma\text{-}alg(\mathcal{T}), \quad \text{and} \quad \mathrm{Baire}(Y) := \sigma\text{-}alg\big(\mathcal{Z}(Y)\big).$$

Remark 2.4. For $f \in C(Y, \mathbb{R})$ we have $Z_n := \{y \in Y\colon\ (|f(y)| \geqslant \frac{1}{n})\} \in \mathcal{Z}(Y)$ and $\mathrm{coz}(f) = \bigcup_n Z_n$, so $\sigma\text{-}lat(\mathcal{Z}) = \sigma\text{-}alg(\mathcal{Z}) = \mathrm{Baire}(Y)$.

When $\mathcal{B} \subseteq \mathcal{P}(Y)$, the $\sigma\text{-}lat(\mathcal{B})$ may be constructed "from the inside out" by recursion on the ordinals $< \omega^+$ as follows:

$$\mathcal{B}_0 := \mathcal{B};$$

$$\mathcal{B}_\eta := \bigcup_{\xi < \eta} \mathcal{B}_\xi \quad \text{for limit ordinals } \eta < \omega^+; \quad \text{and}$$

$$\mathcal{B}_{\eta+1} := \mathcal{B}_\eta \cup \left\{ \bigcap_n B_n\colon\ B_n \in \mathcal{B}_n \right\} \cup \left\{ \bigcup_n B_n\colon\ B_n \in \mathcal{B}_n \right\} \quad \text{for each } \eta < \omega^+.$$

It is clear with these definitions that $\sigma\text{-}lat(\mathcal{B}) = \bigcup_{\eta < \omega^+} \mathcal{B}_\eta$.

It is useful to notice that if κ is an infinite cardinal and $\mathcal{B}$ a family of sets such that $|\mathcal{B}| \leqslant \kappa$, then $|\sigma\text{-}lat(\mathcal{B})| \leqslant \kappa^\omega$. Indeed, inductively one has $|\mathcal{B}_\eta| \leqslant \kappa^\omega$ for each $\eta < \omega^+$, so $|\sigma\text{-}lat(\mathcal{B})| \leqslant \kappa^\omega \cdot \omega^+ = \kappa^\omega$. Here is a consequence of that observation.

Corollary 2.5. *Let Y be a separable space. Then* $|\mathrm{Baire}(Y)| \leqslant \mathfrak{c}$.

With that preliminary material in hand, we can move to the subject-matter proper of this section. Here, the formulations of Theorems 2.8 and 2.11 profit mightily from suggestions provided by Norman Noble: In an earlier version of this paper, circulated for comment to several knowledgeable friends and colleagues, I had speculated on the truth of these results as they now appear, but I had proved Theorem 2.8 only for Hausdorff spaces Y, and Theorem 2.11 only for Tychonoff spaces Y_n. I am grateful to Dr. Noble for permission to adopt, adapt and present his insights here. Some extrapolations of these results are planned in [8].

Our point of departure is a stand-alone topological theorem given in the Henriksen, Isbell and Johnson paper [16], a work devoted principally to quotient fields. With minor changes in notation the statement in question, Lemma 2.2 of [16], reads as follows: *Let X be a subspace of a compact (Hausdorff) space Y such that for some countable family $\mathcal{F}$ of closed subsets of Y, for every pair of points $x \in X$, $y \in Y\backslash X$, there is a set in $\mathcal{F}$ containing x but not y. Then X is a Lindelof space.* It struck me that the lemma was interesting in its own right, and susceptible to generalization. In [16] and in the later expository treatments

[9, (9.7)] and [7], the overlying space Y is assumed to be compact Hausdorff (with $\mathcal{F}$ then defined to be the set of closed subsets of Y).

Now we give a definition and we define notation to be used throughout this section. The former is strictly set-theoretic, topology playing no role.

Definition 2.6. Let $X \subseteq Y$ and $\mathcal{F} \subseteq \mathcal{P}(Y)$. Then $\mathcal{F}$ *distinguishes* X *in* Y if for every pair $(x, y) \in X \times (Y \backslash X)$ there is $F \in \mathcal{F}$ such that $x \in F$, $y \notin F$.

Notation 2.7. For a topological space Y, we write

$$\mathcal{C}(Y) := \{F \subseteq Y\colon\ F \text{ is closed in } Y \text{ and compact in the inherited topology}\}.$$

Theorem 2.8. *Let $Y = (Y, \mathcal{T})$ be a space and let $X \subseteq Y$. If some countable $\mathcal{F} \subseteq \mathcal{C}(Y)$ distinguishes X in Y, then X is a Lindelof space.*

Proof. The proof of Lemma 2.2 of [16] applies with only those minimal changes necessary to accommodate to the lack of hypothesized separation properties. Here are the details.

The condition that $\mathcal{F}$ distinguishes X in Y is equivalent to the condition that for each $x \in X$ there is a family $\mathcal{F}(x) \subseteq \mathcal{F}$ such that $x \in \bigcap \mathcal{F}(x) \subseteq X$. The space $\bigcap \mathcal{F}(x)$ is closed and compact, so if $X \subseteq \bigcup \mathcal{U}$ with $\mathcal{U} \subseteq \mathcal{T}$ then, since $\bigcap \mathcal{F}(x) \subseteq \bigcup \mathcal{U}$, there is (for each $x \in X$) a finite subfamily $\widetilde{\mathcal{F}(x)}$ of $\mathcal{F}(x)$ such that $\bigcap \widetilde{\mathcal{F}(x)} \subseteq \bigcup \mathcal{U}$; so there is a finite subfamily $\mathcal{U}(x)$ of $\mathcal{U}$ such that $x \in \bigcap \widetilde{\mathcal{F}(x)} \subseteq \bigcup \mathcal{U}(x)$. Since $\mathcal{F}$ has only countably many finite subfamilies, there are only countably many sets of the form $\bigcap \widetilde{\mathcal{F}(x)}$. If $\{\bigcap \widetilde{\mathcal{F}(x_k)}\colon\ k < \omega\}$ enumerates those, then $\bigcup \{\mathcal{U}(x_k)\colon\ k < \omega\}$ is a countable subfamily of $\mathcal{U}$ which covers X. $\square$

Remarks 2.9. (a) For some results in the spirit of Theorem 2.8, generalized to higher cardinals, and some applications, see [3, 1.4(iv)–(vi)] and [4, proof of 4.6].

(b) As presently configured, the anticipated work [8] develops the theme initiated in Theorem 2.8 by considering covers of spaces $X \subseteq Y$ by subsets of Y drawn from families different from $\mathcal{C}(Y)$, with applications to unions, intersections, products and some other derived spaces.

We note for emphasis that in the two following theorems, as in Theorem 2.8, the spaces Y and Y_n are not subject to any separation properties whatever.

Theorem 2.10. *Let Y be a space. Then every $X \in \sigma\text{-}lat(\mathcal{C}(Y))$ is a Lindelof space.*

Proof. We write $\mathcal{C} := \mathcal{C}(Y)$. Since $\sigma\text{-}lat(\mathcal{C}) = \bigcup_{\eta < \omega^+} \mathcal{C}_\eta$, it suffices to show this statement for each $\eta < \omega^+$:

$^*(\eta)$ If $X \in \mathcal{C}_\eta$, then some countable $\mathcal{F} \subseteq \mathcal{C}$ distinguishes X in Y.

Statement $^*(0)$ is clear (take $\mathcal{F} = \{X\}$). To prove $^*(\eta + 1)$ assuming $^*(\eta)$, let $X \in \mathcal{C}_{\eta+1}$. There are $X_n \in \mathcal{C}_\eta$ such that $X = \bigcap_n X_n$ or $X = \bigcup_n X_n$ and there are countable families $\mathcal{F}_n \subseteq \mathcal{C}(Y)$ such that $\mathcal{F}_n$ distinguishes X_n in Y. Then $\mathcal{F} := \bigcup_n \mathcal{F}_n$ distinguishes X in Y. $\square$

Theorem 2.11. *For $n < \omega$ let $Y_n = (Y_n, \mathcal{T}_n)$ be a space and let $X_n \in \sigma\text{-}lat(\mathcal{C}(Y_n))$. Then $X := \prod_{n<\omega} X_n$ is a Lindelof space.*

Proof. It suffices, by Theorem 2.10, to find a space $\widetilde{Y}$ such that $X \in \sigma\text{-}lat(\mathcal{C}(\widetilde{Y}))$.

If $(Y_n, \mathcal{T}_n)$ is compact we set $(\widetilde{Y}_n, \widetilde{\mathcal{T}_n}) := (Y_n, \mathcal{T}_n)$. If $(Y_n, \mathcal{T}_n)$ is not compact we set $\widetilde{Y}_n := Y_n \cup \{p_n\}$ with $p_n \notin Y_n$ and we give $\widetilde{Y}_n$ the topology $\widetilde{\mathcal{T}_n} := \mathcal{T} \cup \{U \subseteq \widetilde{Y}_n\colon\ p_n \in U, Y \backslash U \in \mathcal{C}(Y_n)\}$. Then each space $(\widetilde{Y}_n, \widetilde{\mathcal{T}_n})$ is compact, and Y_n is open and $\widetilde{\mathcal{T}_n}$-dense in $\widetilde{Y}_n$.

Give $\widetilde{Y} := \prod_n \widetilde{Y}_n$ the product topology and for $n < \omega$ set $X'_n := X_n \times \prod_{m \neq n} \widetilde{Y}_m$. From $X_n \in \sigma\text{-}lat(C(Y_n))$ and the fact that $\prod_{m \neq n} \widetilde{Y}_m$ is compact it follows that $X'_n \in \sigma\text{-}lat(\mathcal{C}(\widetilde{Y}))$. Then $X = \bigcap_n X'_n \in \sigma\text{-}lat(\mathcal{C}(\widetilde{Y}))$, as required. $\square$

Remarks 2.12. (a) It is immediate from Theorem 2.11 that for every space Y and $X \in \mathcal{C}(Y)$ the space X^ω is a Lindelof space. Further, every product of countably many σ-compact spaces is a Lindelof space; for Hausdorff spaces this latter fact has been noted by Frolík [11, Theorem 10] and Hager [13].

(b) It is interesting to note that a σ-compact Hausdorff space, though clearly Lindelof, need not be a Tychonoff space. The exposition and discussion given in [21, (#64)] of “Smirnov's deleted sequence topology” on $\mathbb{R}$ illuminates an example.

(c) In a non-Hausdorff space Y, a subset X distinguished by countably many compact subsets need not be a Lindelof space. Let $X = (X, \mathcal{T})$ be an arbitrary (possibly non-Lindelof) space and define $(Y, \mathcal{U})$ so that $Y := X \cup \{p_0, p_1\}$ and $\mathcal{U} := \mathcal{T} \cup \{V \subseteq Y\colon\ |Y \backslash V| < \omega\}$. Let $F_i := Y \backslash \{p_i\}$ for $i = 0, 1$ and set $\mathcal{F} := \{F_0, F_1\}$. Then the elements of $\mathcal{F}$ are compact,

$\mathcal{F}$ distinguishes X in Y, and $F_0 \cap F_1 = X$. Since $\mathcal{T} \subseteq \mathcal{U}$, X is not Lindelof in the topology inherited from $(Y, \mathcal{U})$ if $(X, \mathcal{T})$ is not Lindelof. (Note further in this case that if $X = (X, \mathcal{T})$ was chosen to be a T_1 space, then $Y = (Y, \mathcal{U})$ also is a T_1 space.)

Corollary 2.13. *Let Y be a space and let $X \in \text{Baire}(Y)$. Then*

(a) *if Y is a σ-compact Hausdorff space then X is a Lindelof space*;
(b) *if Y is realcompact then X is realcompact*;
(c) *if Y is topologically complete then X is topologically complete*; *and*
(d) *if Y is a Tychonoff space and $Y \in \text{Baire}(\beta Y)$ then X is a Lindelof space.*

Proof. (a) Let $Y = \bigcup_n Y_n$ with each Y_n compact. Since Y_n is a Hausdorff space, each set $Z \in \mathcal{Z}(Y_n)$ is closed and compact, and using Remark 2.4 we have

$$X \cap Y_n \in \text{Baire}(Y_n) = \sigma\text{-}alg\big(\mathcal{Z}(Y_n)\big) = \sigma\text{-}lat\big(\mathcal{Z}(Y_n)\big) \subseteq \sigma\text{-}lat\big(\mathcal{C}(Y_n)\big).$$

Then $X \cap Y_n$ is Lindelof by Theorem 2.10, so $X = \bigcup_n (X \cap Y_n)$ is Lindelof.

(b) and (c) Y is a Tychonoff space, and there is $X' \in \text{Baire}(\beta Y)$ such that $X = X' \cap Y$. The space X' is Lindelof by (a), hence realcompact and hence topologically complete, and (b) and (c) are immediate (see [12, (8.9)]).

(d) In this case $X \in \text{Baire}(\beta Y)$, so (a) applies. □

I find the following generalization of Theorem 2.11 a bit surprising, since it indicates a sort of cross-cultural communal behavior among Baire sets in unrelated spaces.

Corollary 2.14. *For $n < \omega$ let Y_n be a σ-compact Hausdorff space and let $X_n \in \text{Baire}(Y_n)$. Then $X := \prod_n X_n$ is a Lindelof space.*

Proof. Again by Remark 2.4 we have $X_n \in \sigma\text{-}lat(\mathcal{Z}(Y_n)) \subseteq \sigma\text{-}lat(\mathcal{C}(Y_n))$ for each $n < \omega$, and Theorem 2.11 applies. □

Remarks 2.15. (a) In constructing $\sigma\text{-}lat(\mathcal{B})$ from a family $\mathcal{B} \subseteq \mathcal{P}(Y)$ as above, one passes from $\mathcal{B}_\eta$ to $\mathcal{B}_{\eta+1}$ by adjoining to $\mathcal{B}_\eta$ both the union and intersection of each of its countable subfamilies. I find it amusing, though of a significance I have not discerned, to notice that in Theorem 2.10 and Corollary 2.13(a) and (d) those unions trivially have the properties in question but the assertion concerning intersection is much less obvious; while in Corollary 2.13(b) and (c) it is just the opposite: the intersection statements are nearly trivial (since in any space the intersection of any family of realcompact [resp., topologically complete] subsets is realcompact [resp., topologically complete]), while the assertions concerning unions are not obvious.

(b) The relation "is Baire in", though transitive in the specific instance considered in Corollary 2.13(d), is in general not transitive. For a simple example to this effect, let D be a discrete subset of $\beta\mathbb{N} \backslash \mathbb{N}$ such that $|D| = \mathfrak{c}$, as given for example in [12, (6.Q.4)], and set $Y := \mathbb{N} \cup D$. Then $\text{Baire}(D) = \mathcal{P}(D)$ and $D \in \mathcal{Z}(Y) \subseteq \text{Baire}(Y)$. But $\text{Baire}(D) \subseteq \text{Baire}(Y)$ fails since $|\text{Baire}(Y)| = \mathfrak{c}$ (cf. Corollary 2.5 above), while $|\text{Baire}(D)| = |\mathcal{P}(D)| = 2^{\mathfrak{c}}$.

(c) It is known (in ZFC) that there exist a paracompact (Tychonoff) space Y with $X \in \text{Baire}(Y)$ such that X is not paracompact. See [7, (11.8)] for an example. The argument given there shows under [CH] that there are a Lindelof space Y and $X \in \text{Baire}(Y)$ such that X is not Lindelof, but the authors of [7] were not able to determine whether such an example exists in ZFC.

Remark 2.16. The paper of Frolík [11], while differing in emphasis from ours in this work, introduces and discusses several "countably multiplicative" classes of spaces. To avoid straying too far from the paper [16], I have not pursued the relation between Frolík's classes and those discussed here. Of particular relevance (see [11, Theorem 12]) is the class of spaces homeomorphic to a closed subspace of a space of the form $\prod_n Y_n$ with each Y_n σ-compact and regular.

Before moving on, I cannot resist the temptation to discuss briefly a natural question. Recall that, given a space Y, the σ-algebra of Baire sets may be realized in the form $\text{Baire}(Y) = \bigcup_{\eta<\omega^+} \mathcal{Z}_\eta$, where $\mathcal{Z} = \mathcal{Z}_0 = \mathcal{Z}(Y)$. Suppose that, as you move through the ordinals $\eta < \omega^+$, you stumble across a closed set $F \in \mathcal{Z}_\eta$. Must necessarily F have been present all along—that is, must $F \in \mathcal{Z} = \mathcal{Z}_0$? An appropriate response here might take the form "Often Yes, but sometimes No". I believe that Halmos was the first to consider questions of this kind; he showed in [14, (51D)] that every compact Baire set F, in every space Y, is necessarily a zero-set. Speaking informally, we may say that the proof consists in finding a metric space M and a surjective perfect map $f : Y \twoheadrightarrow M$ such that $F = f^{-1}(f[F])$. Essentially the same proof works for every closed $F \in \text{Baire}(Y)$ when $Y \in \text{Baire}(\beta Y)$. The papers [20] and [5] broaden our knowledge of those Tychonoff spaces X for which every closed $F \in \text{Baire}(Y)$ is a zero-set, but so far as I am aware the complete classification of those spaces has not been determined. In general, a closed Baire set need not be a zero-set. For an example, defined by Katětov [17] for a related purpose, it is enough to take $Y := \beta\mathbb{R} \backslash (\beta\mathbb{N} \backslash \mathbb{N})$ and $F := \mathbb{N} \subseteq Y$; see [12, (6.P.5)] for a detailed argument.

In this article honoring Mel Henriksen and his achievements, I do not want to make the emotional or illogical error of crediting to Mel and his co-authors the many fine results I have cited here which are provable on the basis of perfect functions, or which follow with a little effort from the Henriksen–Isbell–Johnson topological lemma. But it does seem fair to assert that without those two papers, these several theorems would not have become available.

3. The paper [1]

Now I shift gears a bit, moving about 35 years ahead. In July, 1996 I had an e-mail from Mel growing peripherally out of some collaborative work under way with Grant Woods. Here is that e-mail in its entirety.

From: HENRIKSEN@THUBAN.AC.HMC.EDU
To: WCOMFORT@EAGLE.WESLEYAN.EDU
Sent: 8:46P.M. 7/29/1996
Suppose X and Y are Tychonoff spaces.
Is it true that Card(C(XxY)) = Card(C(X))Card(C(Y))? This clearly holds if X and Y are compact. Is the answer known in the general case?
Mel

Those who knew him are well aware that Mel could be a very social animal—polite, thoughtful, caring. But this communication was Mel at his best in full battle mode. In July, 1996 we hadn't seen or heard from each other in many months, but here's a memo from the blue with no salutation or greeting, no chit-chat about our last pleasant meal in San Antonio, no hopes for Mary Connie's good health. Just basically: Let's get on this, it's bugging me and it looks like fun. A couple of days later I made a few remarks, intended to be helpful, and those in turn prompted this observation by e-mail from Mel, derived again I believe in conversations with Grant Woods.

Theorem 3.1. *There are a Tychonoff space X and a discrete space D such that $|C(X \times D)| > |C(X)| \cdot |C(D)|$ if and only if there are cardinals m and t such that $m^t > m^\omega = m \geqslant 2^t$.*

Since it's part of the story, let's verify the "if" part there. (The "only if" need not concern us in this narrative.) Given such m, let X be a Tychonoff space such that $|C(X)| = m$, for example, let $X = \{0,1\}^m$ or $X = [0,1]^m$, and let D be discrete with $|D| = t$. Then $|C(X)| = m = m^\omega$ and $|C(D)| = 2^t$, while clearly $|C(X \times D)| = m^t$, with $m^t > m^\omega$ and $m^t > 2^t$, so

$$\left|C(X \times D)\right| = m^t > m^\omega \cdot 2^t = \left|C(X)\right| \cdot \left|C(D)\right|.$$

As it happens, that exact condition on (m,t) had appeared in a paper written some years earlier by Tony Hager and me [6], also about conditions on cardinal numbers of the form $|C(X)|$, which explains why Mel wrote inquiring about the existence of such pairs. It is embarrassing to report that Tony and I in our paper had left unresolved the existence of such pairs in ZFC, but in fact the existence of many such pairs is easily established. There is no time for the details now, but any reader familiar with the vocabulary will see quickly that, beginning with any cardinal t of uncountable cofinality such that $t < \beth_t$ it is enough to take $m := \beth_{\aleph_1}(t)$. (Note: the smallest such pair in ZFC then is $(m,t) = (\beth_{\aleph_1}, \aleph_1)$.) So, Yes, the obvious inequality $|C(X \times Y)| \geqslant |C(X)| \cdot |C(Y)|$ is strict in many cases. But as always, a myriad of unanswered questions arose in the wake of that observation. Mel appointed, convened, coordinated and chaired a committee to address these issues. This was my first and only collaboration in research with Mel, conducted electronically on my part, in the company of Ofelia Alas, Salvador Garcia-Ferreira, Richard Wilson and Grant Woods. For over a year I happily pictured Mel at his Command Central Headquarters in Claremont, California, fielding conjectures and raw theorems from his troops, sometimes incorrectly formulated and proved, firing back an occasional word of satisfaction followed by a goading question, always respectful but sometimes with a hint of impatience, all the while integrating, shaping and massaging our missives into something resembling a coherent research paper—and always, needless to say, improving our results and adding more of his own. The paper [1] contains much material. I here list only the two simplest observations given in [1], and some attractive questions left unanswered there.

Definition 3.2. (a) A pair (X,Y) of spaces is *functionally conservative* (henceforth: f.c.) if $|C(X \times Y)| = |C(X)| \cdot |C(Y)|$; and (b) Y is *f.c.* if (X,Y) is f.c. for all X.

Theorem 3.3. *Always $|C(X \times Y)| \leqslant |C(X)|^{d(Y)}$.*

Proof. For D dense in Y the map $C(X \times Y) \to C(X \times D)$ given by $f \to f|(X \times D)$ is injective, so $C(X \times Y)| \leqslant |C(X \times D)| \leqslant |(C(X))^D|$. □

Corollary 3.4. *Every separable space is f.c.*

Proof. It is easily seen for every space X that $|C(X)| = |C(X)|^\omega$, so Theorem 3.3 shows that $|C(X \times Y)| \leqslant |(C(X))^\omega| = |C(X)|$ when Y is separable. □

It is not difficult to see that the product of finitely many f.c. spaces (in fact, of countably many) is f.c. But by (the proof of) Theorem 3.1 the space $\{0,1\}^{\beth_{\aleph_1}}$ is not f.c.

Question 3.5. (a) What is the least κ such that some product of κ-many f.c. spaces is not f.c.?

(b) Is there an f.c. space X such that X^κ is not f.c.?

Concerning Question 3.5(a) we note that $\mathrm{cf}(\kappa) > \omega$, so $\aleph_1 \leqslant \kappa \leqslant \beth_{\aleph_1}$, $\kappa \neq \aleph_\omega, \kappa \neq \beth_\omega$, etc.

One checks easily that the space $[0, \aleph_1]$ is f.c., but $[0, \beth_{\aleph_1}]$ is not. Therefore we ask [1]:

Question 3.6. What is the least cardinal λ such that $[0, \lambda]$ is not f.c.?

It is known [1] that an f.c. space need not be separable, but the examples X given in [1] all satisfy $|C(X)| = 2^\omega = \mathfrak{c}$. So we ask [1]:

Question 3.7. Is there an f.c. space such that $|C(X)| > 2^\omega = \mathfrak{c}$?

4. Concluding comment

As noted in footnote 1, this article grew out of an address delivered March 27, 2010 to a conference honoring Mel Henriksen at Harvey Mudd College in California. In preparing those remarks I spent much time living in a sense with Mel, reviewing the papers [15,16] and [1] and others and re-reading our extensive personal correspondence, much of which pre-dates the electronic era. I feel I have come to know Mel much better than was the case before. I have come to perceive his work not only as incisive and definitive, but also as fertile and open–ended. It pleases me, and somehow I think it would please Mel, too, to recognize that, as I have attempted to show here, his work is both polished and complete, while simultaneously generating and provoking intriguing questions which assure the continuation of his legacy and influence into the future.

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