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ScienceDirect
Topology and its Applications 154 (2007) 205–214
Topology and its Applications
www.elsevier.com/locate/topol

Resolvability properties via independent families

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Received 20 October 2005; accepted 9 February 2006

Abstract

The authors give a consistent affirmative response to a question of Juhász, Soukup and Szentmiklóssy: If GCH fails, there are (many) extraresolvable, not maximally resolvable Tychonoff spaces. They show also in ZFC that for $\omega < \lambda \leqslant \kappa$, no maximal λ-independent family of λ-partitions of κ is ω-resolvable. In topological language, that theorem translates to this: A dense, ω-resolvable subset of a space of the form $(D(\lambda))^I$ is λ-resolvable.

MSC: primary 05A18, 03E05, 54A10; secondary 03E35, 54A25, 05D05

Keywords: Extraresolvable space; Maximally resolvable space; ω-resolvable space; Souslin number; Independent family

1. Introduction and historical perspective

Continuing earlier projects [20,6,21,7] we again use combinatorial principles (the existence of large families of sets with independence properties) to construct spaces with properties of resolvability type. In parts (A) and (B) of this introductory section we give the necessary notation and background concerning resolvability and independent families, respectively.

(A) Properties of resolvability type

According to terminology introduced by Hewitt [19] in 1943, a topological space is *resolvable* if it contains complementary dense subsets. In subsequent decades, several variations on Hewitt's concept appeared in the literature. Throughout, for a space $X = \langle X, \mathcal{T}\rangle$ we denote by $\Delta(X)$ the number $\Delta(X) = \min\{|U|\colon \emptyset \neq U \in \mathcal{T}\}$, and we denote by $\mathrm{nwd}(X)$ the *nowhere density number* of X, defined by the relation $\mathrm{nwd}(X) = \min\{|A|\colon A \subseteq X,\ \mathrm{int}_X \overline{A}^X \neq \emptyset\}$. (Evidently $\mathrm{nwd}(X)$ coincides with the so-called *open density* number $\mathrm{od}(X)$ preferred by some authors; this is defined by the relation $\mathrm{od}(X) = \min\{d(U)\colon \emptyset \neq U \in \mathcal{T}\}$.)

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0166-8641/$ – see front matter
doi:10.1016/j.topol.2006.02.005

Definitions 1.1. A topological space $X = \langle X, \mathcal{T}\rangle$ is

(i) [2] *κ-resolvable* if X admits a collection of κ-many pairwise disjoint $\mathcal{T}$-dense subsets;
(ii) [2] *maximally resolvable* if X is $\Delta(X)$-resolvable;
(iii) [26] *extraresolvable* if X admits a collection $\mathcal{D}$ of $\mathcal{T}$-dense subsets, with $|\mathcal{D}| = (\Delta(X))^+$, such that: If D_0, D_1 are distinct elements of $\mathcal{D}$ then $D_0 \cap D_1$ is nowhere dense in X; and
(iv) [5] *strongly extraresolvable* if X admits a collection $\mathcal{D}$ of $\mathcal{T}$-dense subsets, with $|\mathcal{D}| = (\Delta(X))^+$, such that: If D_0, D_1 are distinct elements of $\mathcal{D}$ then $|D_0 \cap D_1| < \mathrm{nwd}(X)$.

Discussion 1.2. Many authors have investigated the relations among these concepts. The following brief review of the results most relevant to the present investigation will help to orient the reader.

(a) For $0 < n < \omega$ there is a Tychonoff space which is n-resolvable but not $(n+1)$-resolvable [10]; other examples, not all Tychonoff, are given in [12,15,11,16].
(b) A space which is n-resolvable for each $n < \omega$ is ω-resolvable [22]; the natural generalization to cardinals $\kappa > \omega$ with $\mathrm{cf}(\kappa) = \omega$ was given subsequently in [1].
(c) The question whether every ω-resolvable space is maximally resolvable, dating from 1967 [3], proved unexpectedly elusive. Over the years, examples responding in the negative were given [13,25,11,20,21], but these were viewed as mildly unsatisfactory in that each was either non-Tychonoff or was a "consistent example", i.e., a space defined when ZFC is augmented by some additional axiom(s). The existence in ZFC of many Tychonoff examples has recently been established by Juhász, Soukup, and Szentmiklóssy [24].
(d) The question of the existence of a countable Tychonoff extraresolvable space which is not strongly extraresolvable, raised in [4], was answered affirmatively in [18]. Subsequently the present authors [7] gave for each $\kappa \geqslant \omega$ a Tychonoff space $X = X(\kappa)$, simultaneously maximally resolvable and extraresolvable, such that $|X| = \mathrm{nwd}(X) = \kappa$ and X is not strongly extraresolvable.

In this paper, using an enhanced version of techniques introduced earlier [20,6,21,7], we give in Theorem 3.1 and its corollaries a consistent negative response to a question posed as Problem 1.1 in [24]: Is an extraresolvable Tychonoff space necessarily maximally resolvable? Our argument requires the failure of GCH, and the stated question remains open in ZFC.

[Note added March 24, 2006. We learned today that an example responding to Problem 1.1 in [24] has been given in ZFC by Juhász, Shelah and Szentmiklóssy.]

(B) Independent families

We use α, β, γ, ξ and η to denote ordinals, while κ, λ, μ and τ denote cardinals. We set $[\kappa]^\lambda := \{A \subseteq \kappa\colon |A| = \lambda\}$; the notations $[\kappa]^{\leqslant\lambda}$ and $[\kappa]^{<\lambda}$ are defined analogously. For sets A and B we denote by $Fn(A, B)$ the set of functions from a finite subset of A into B. In symbols: $Fn(A, B) := \bigcup\{{}^F B\colon F \in [A]^{<\omega}\}$.

$D(\lambda)$ denotes the set λ with the discrete topology.

Definition 1.3. Let $2 \leqslant \lambda \leqslant \kappa$ and $\kappa \geqslant \omega$.

(a) A *λ-partition* of κ is a partition of κ into λ-many pairwise disjoint nonempty sets;
(b) a family $\mathbf{B} = \{\mathcal{B}_i\colon i \in I\}$ of (faithfully indexed) λ-partitions $\mathcal{B}_i = \{B_i^\eta\colon \eta < \lambda\}$ of κ is *τ-independent* if

$$\varepsilon \in Fn(I, \lambda) \implies \left|\bigcap\left\{B_i^{\varepsilon(i)}\colon i \in \mathrm{dom}(\varepsilon)\right\}\right| \geqslant \tau.$$

For notational simplicity, given κ, λ, I and $\mathbf{B}$ as above, we write

$$\mathbf{B}(\varepsilon) := \bigcap\left\{B_i^{\varepsilon(i)}\colon i \in \mathrm{dom}(\varepsilon)\right\} = \left\{x \in \kappa\colon i \in \mathrm{dom}(\varepsilon) \Longrightarrow x \in B_i^{\varepsilon(i)}\right\}.$$

Discussion 1.4. We use the notation $\mathcal{T}_\mathbf{B}$ to denote the (smallest) topology on κ for which each set $B_i^\eta \in \mathcal{B}_i \in \mathbf{B}$ is open; clearly each such B_i^η is $\mathcal{T}_\mathbf{B}$-closed, and $\{\mathbf{B}(\varepsilon)\colon \varepsilon \in Fn(I, \lambda)\}$ is a basis for $\mathcal{T}_\mathbf{B}$. This is a Hausdorff topology if and only if $\mathbf{B}$ *separates points* of κ in the sense that for distinct $x, x' \in \kappa$ there are $\mathcal{B}_i \in \mathbf{B}$ and (distinct) $\eta, \eta' < \lambda$ such that $x \in B_i^{\eta}$ and $x' \in B_i^{\eta'}$; when this occurs the space $\langle \kappa, \mathcal{T}_{\mathbf{B}} \rangle$ is a Tychonoff space—indeed the evaluation map $e_{\mathbf{B}} : \langle \kappa, \mathcal{T}_{\mathbf{B}} \rangle \to (D(\lambda))^I$ given by

$$(e_{\mathbf{B}} x)_i = \eta \quad \text{if } x \in B_i^{\eta} \quad (x \in \kappa,\ i \in I,\ \eta < \lambda)$$

is a topological embedding of $\langle \kappa, \mathcal{T}_{\mathbf{B}} \rangle$ onto a subspace X of the Tychonoff space $(D(\lambda))^I$; here the range $X := e_{\mathbf{B}}[\kappa]$ is dense in $(D(\lambda))^I$ iff $\mathbf{B}$ is 1-independent. In this paper in this context, λ and κ being given, we use the notations $\langle \kappa, \mathcal{T}_{\mathbf{B}} \rangle$, X and $e_{\mathbf{B}}[\kappa]$ interchangeably.

(In the proof of Theorem 3.1 below we will use the fact that on a set of infinite cardinality τ there exists for $2 \leqslant \lambda \leqslant \tau$ a τ-independent family $\mathbf{I} = \{\mathcal{I}_\gamma \colon \gamma < 2^\tau\}$ of λ-partitions $\mathcal{I}_\gamma = \{I_\gamma^\beta\}$ which is even *small-set-separating* in the sense that for disjoint $S, S' \in [\tau]^{<\tau}$ there are $\gamma < 2^\tau$ and $\eta, \eta' < \lambda$ such that $S \subseteq I_\gamma^\beta$ and $S' \subseteq I_\gamma^{\eta'}$. A routine argument proving that fact, exploiting a trick introduced by Eckertson [11] in a related context, is given in [6] and in [7, 3.3(b)].)

In the converse direction, given $2 \leqslant \lambda \leqslant \kappa$ and $\kappa \geqslant \omega$, the existence of a point-separating, 1-independent family $\mathbf{B}$ of λ-partitions of κ of maximal cardinality $|\mathbf{B}| = 2^\kappa$ is given by the following version of the familiar Hewitt–Marczewski–Pondiczery theorem. (A succinct statement and proof of this theorem are available in [14, 2.3.15]. See also [8, §3 and its Notes] for detailed bibliographic references and for applications of "families of large oscillation" to the calculation of the density character of various product spaces in modified box topologies.)

Theorem 1.5. *Let $2 \leqslant \lambda \leqslant \kappa$ and $\kappa \geqslant \omega$. Then $d((D(\lambda))^{2^\kappa}) = \log(2^\kappa) \leqslant \kappa$.*

To derive the existence of such a family $\mathbf{B}$ from Theorem 1.5 it is enough, given X dense in $(D(\lambda))^{2^\kappa}$ with $|X| = \kappa$, to define $\mathbf{B} := \{\mathcal{B}_\alpha \colon \alpha < 2^\kappa\}$ with $\mathcal{B}_\alpha = \{B_\alpha^\eta \colon \eta < \lambda\}$, $B_\alpha^\eta = \{x \in X \colon x_\alpha = \eta\}$.

In what follows, we will want to have the family $\mathbf{B}$ and the dense set $X = e_{\mathbf{B}}[\kappa] = \langle \kappa, \mathcal{T}_{\mathbf{B}} \rangle \subseteq (D(\lambda))^{2^\kappa}$ so that some properties beyond those given by the bare-bones Hewitt–Marczewski–Pondiczery theorem are satisfied. Specifically, we want:

(i) $\mathbf{B}$ is not only 1-independent, but κ-independent;
(ii) the space $\langle X, \mathcal{T}_{\mathbf{B}} \rangle$ is κ-resolvable; and
(iii) $\mathrm{nwd}\langle X, \mathcal{T}_{\mathbf{B}} \rangle = \kappa$.

To arrange this, again given $2 \leqslant \lambda \leqslant \kappa$ and $\kappa \geqslant \omega$, begin with X dense in $(D(\lambda))^{2^\kappa}$ and $|X| = \kappa$, and give $D(\lambda)$ the structure of a (discrete) topological group. Let $\langle X \rangle$ be the subgroup of $(D(\lambda))^{2^\kappa}$ generated by X, and let $\widetilde{X}$ be the union of κ-many (disjoint) translates of $\langle X \rangle$. Next, much as in [7, 3.8], write $\sigma := \{p \in (D(\lambda))^\kappa \colon |\{\alpha < \kappa \colon p_\alpha \neq 0_\alpha\}| < \omega\}$ and define $X^* := \widetilde{X} \times \sigma \subseteq (D(\lambda))^{2^\kappa} \times (D(\lambda))^\kappa \simeq (D(\lambda))^{2^\kappa}$. It is easy to see that $\mathrm{nwd}(\sigma) = \kappa$, so $\kappa = |X^*| \geqslant \mathrm{nwd}(X^*) = \mathrm{nwd}(\widetilde{X}) \times \mathrm{nwd}(\sigma) \geqslant \mathrm{nwd}(\sigma) = \kappa$, so $|X^*| = \mathrm{nwd}(X^*) = \kappa$.

Identifying X^* with κ, we summarize, emphasizing that the following statement is simply a (slightly redundant) enhanced restatement of Theorem 1.5.

Theorem 1.6. *Let $2 \leqslant \lambda \leqslant \kappa$ and $\kappa \geqslant \omega$. There is a κ-independent, point-separating family $\mathbf{B}$ of λ-partitions of κ such that $|\mathbf{B}| = 2^\kappa$, $\mathrm{nwd}\langle \kappa, \mathcal{T}_{\mathbf{B}} \rangle = \kappa$ and the space $\langle \kappa, \mathcal{T}_{\mathbf{B}} \rangle$ is κ-resolvable and homeomorphic to a dense subspace of the space $(D(\lambda))^{2^\kappa}$.*

Not every κ-independent family $\mathbf{B}$ of λ-partitions of κ induces on κ a topology $\mathcal{T}_{\mathbf{B}}$ which is κ-resolvable. The following simple observation, based on the fact that every κ-independent family $\mathbf{B}$ of λ-partitions of κ expands *via* Zorn's Lemma to a maximal such partition, will be useful later. Here we say as usual that a subset D of a space X is *κ-dense* in X if $|D \cap U| \geqslant \kappa$ for each (nonempty, basic) open subset U of X. We note that a set dense in a space X with $\mathrm{nwd}(X) = \kappa$ is itself necessarily κ-dense.

Theorem 1.7. *Let $2 \leqslant \lambda \leqslant \kappa$ and $\kappa \geqslant \omega$, and let $\mathbf{B}$ be a point-separating κ-independent family of λ-partitions of κ. Then the following conditions are equivalent*:

(a) *κ can be partitioned into λ-many subsets, each κ-dense in* $(\kappa, \mathcal{T}_{\mathbf{B}})$; *and*
(b) *$\mathbf{B}$ is not maximal among κ-independent families of λ-partitions of κ.*

If in addition $\mathrm{nwd}\langle\kappa, \mathcal{T}_{\mathbf{B}}\rangle = \kappa$, *then the following condition is also equivalent to* (a) *and* (b):

(c) *$\langle\kappa, \mathcal{T}_{\mathbf{B}}\rangle$ is λ-resolvable.*

2. Families of partitions with special properties

A result parallel to the following theorem, with a different proof, is given in Main Theorem 3.3 of [24]. Our argument follows the initiative of [20,21], [7, Theorem 3.6]. The objective here is to begin with a family $\mathbf{B}$ of λ-partitions of κ with properties as guaranteed by Theorem 1.6, and to replace $\mathbf{B}$ with a related family $\mathbf{C}$ enjoying more subtle properties. The family $\mathbf{C}$ is defined in terms of a pre-assigned family $\mathcal{D}$ of $\mathcal{T}_{\mathbf{B}}$-dense subsets of κ, whose members are to remain $\mathcal{T}_{\mathbf{C}}$-dense, while certain other sets—for example, any set $E \subseteq \kappa \setminus \bigcup \mathcal{D}$—will emphatically not be $\mathcal{T}_{\mathbf{C}}$-dense in κ.

Here and later, given a space $\langle X, \mathcal{T}\rangle$ and $E \subseteq X$ we denote by $\langle E, \mathcal{T}\rangle$ the set E with the topology inherited from $\langle X, \mathcal{T}\rangle$. We denote by $S(X)$ the *Souslin number* of X, that is, the least cardinal number μ such that X admits no family of μ-many pairwise disjoint nonempty open subsets. It is clear that if E is dense in X, then $S(E) = S(X)$.

Since our principal application of Theorem 2.1 will be in Section 3 below, where at times GCH is assumed violated, we remark for clarity here that 2.1 itself is a theorem of ZFC.

Theorem 2.1. *Let $2 \leqslant \lambda \leqslant \kappa$ and $\kappa \geqslant \omega$, and let $\mathbf{B} = \{\mathcal{B}_\alpha\colon \alpha < 2^\kappa\}$ be a point-separating, κ-independent family of λ-partitions of κ such that* $\mathrm{nwd}\langle\kappa, \mathcal{T}_{\mathbf{B}}\rangle = \kappa$. *Let $\mathcal{D}$ be a set of dense subsets of the space $(\kappa, \mathcal{T}_{\mathbf{B}})$. Then there is a point-separating, κ-independent family $\mathbf{C} = \{\mathcal{C}_\alpha\colon \alpha < 2^\kappa\}$ of λ-partitions of κ such that*

(a) *each $D \in \mathcal{D}$ is κ-dense in $\langle\kappa, \mathcal{T}_{\mathbf{C}}\rangle$;*
(b) *each $E \in [\kappa]^{<\kappa}$ is closed, discrete and nowhere dense in $\langle\kappa, \mathcal{T}_{\mathbf{C}}\rangle$; and*
(c) *each $E \subseteq \kappa$ such that* $\mathrm{int}_{(D,\mathcal{T}_{\mathbf{C}})}(D \cap E) = \emptyset$ *for all $D \in \mathcal{D}$ is closed, discrete and nowhere dense in $\langle\kappa, \mathcal{T}_{\mathbf{C}}\rangle$.*

Proof. We assume the notation chosen so that the initial segment $\{\mathcal{B}_\beta\colon \beta < \kappa\}$ of $\mathbf{B}$ already separates points of κ.

Let $\{K_\alpha\colon \alpha < 2^\kappa\}$ be an enumeration of $\mathcal{P}(\kappa)$, with $K_0 = \emptyset$.

Let $I = 2^\kappa \times \kappa$ be lexicographically ordered and write $\mathbf{B} := \{\mathcal{B}_i\colon i \in I\}$ with $\mathcal{B}_i = \{B_i^\eta\colon \eta < \lambda\}$; in this indexing, we identify the initial segment $\{\mathcal{B}_\beta\colon \beta < \kappa\}$ of $\mathbf{B}$ with the initial segment $\{\mathcal{B}_{(0,\beta)}\colon \beta < \kappa\}$ of $\mathbf{B}$. We define recursively $\mathcal{C}_i$ and a family $\mathbf{A}_i$ of κ-independent λ-partitions as follows.

If $i < (1,0)$ then $\mathbf{A}_i := \mathbf{B}$ and $\mathcal{C}_i := \mathcal{B}_i$, with $C_i^\eta = B_i^\eta$ for $\eta < \lambda$.

Let $(1,0) \leqslant i = (\alpha, \beta) \in I$, and suppose that $\mathbf{A}_{i'}$ and $\mathcal{C}_{i'}$ and have been defined for all $i' < i$. Set

$$\mathbf{A}_i := \{\mathcal{C}_{i'}\colon i' < i\} \cup \{\mathcal{B}_{i'}\colon i' \geqslant i\}.$$

To define $\mathcal{C}_i$, we consider two cases.

Case 1. There are $D \in \mathcal{D}$ and nonempty $U \in \mathcal{T}_{\mathbf{A}_i}$ such that $|(D \cap U) \setminus K_\alpha| < \kappa$. Then $\mathcal{C}_i := \mathcal{B}_i$ with $C_i^\eta = B_i^\eta$ for $\eta < \lambda$.

Case 2. Case 1 fails. Then set

$$C_i^0 = \left(B_i^0 \cup K_\alpha\right) \setminus \{\beta\},$$
$$C_i^1 = \left(B_i^1 \setminus K_\alpha\right) \cup \{\beta\}, \quad \text{and}$$
$$C_i^\eta = B_i^\eta \setminus \left(K_\alpha \cup \{\beta\}\right) \quad \text{for } 2 \leqslant \eta < \lambda.$$

The definitions are complete. Clearly $\mathbf{A}_i$ is a family of λ-partitions of κ, and $\mathcal{C}_i := \{C_i^\eta\colon \eta < \lambda\}$ is a λ-partition of κ. That $\mathbf{C} := \{\mathcal{C}_i\colon i \in I\}$ separates points of κ follows from the corresponding property of $\{\mathcal{B}_\beta\colon \beta < \kappa\} \subseteq \mathbf{B}$.

We show for each $i = (\alpha, \beta)$ that the family $\mathbf{A}_i$ is κ-independent. In fact we show more: $|D \cap U| = \kappa$ for each $D \in \mathcal{D}$ and each (basic) $U \in \mathcal{T}_{\mathbf{A}_i}$.

That is clear for $i < (0,1)$. Now fix $i \geqslant (0,1)$, suppose the statement true for all $i' < i$, and let

$$U = \mathbf{A}_i(\varepsilon) = \bigcap_{i > j \in \mathrm{dom}(\varepsilon)} C_j^{\varepsilon(j)} \cap \bigcap_{i \leqslant j \in \mathrm{dom}(\varepsilon)} B_j^{\varepsilon(j)}$$

be $\mathcal{T}_{\mathbf{A}_i}$-basic (with $\varepsilon \in Fn(I,\lambda)$). If i is a limit in I then since $\mathrm{dom}(\varepsilon)$ is finite there is $i' < i$ such that $U \in \mathcal{T}_{\mathbf{A}_{i'}}$ and the inductive hypothesis applies, so for notational simplicity we write $i = (\alpha, \beta+1)$; then either $(\alpha,\beta) \notin \mathrm{dom}(\varepsilon)$, in which case $U \in T_{\mathbf{A}_{i'}}$ with $i' = (\alpha,\beta) < i$, or $U = \mathbf{A}_i(\varepsilon)$ has the form

$$U = \bigcap_{(\alpha,\beta) > j \in \mathrm{dom}(\varepsilon)} C_j^{\varepsilon(j)} \cap C_{(\alpha,\beta)}^{\varepsilon(\alpha,\beta)} \cap \bigcap_{i \leqslant j \in \mathrm{dom}(\varepsilon)} B_j^{\varepsilon(j)}.$$

Now define

$$V := \bigcap_{(\alpha,\beta) > j \in \mathrm{dom}(\varepsilon)} C_j^{\varepsilon(j)} \cap B_{(\alpha,\beta)}^{\varepsilon(\alpha,\beta)} \cap \bigcap_{i \leqslant j \in \mathrm{dom}(\varepsilon)} B_j^{\varepsilon(j)},$$

and note that $V \in \mathcal{T}_{\mathbf{A}_{(\alpha,\beta)}}$ with $(\alpha,\beta) < i$. If $B_{(\alpha,\beta)}^{\varepsilon(\alpha,\beta)} = C_{(\alpha,\beta)}^{\varepsilon(\alpha,\beta)}$ then $U = V \in \mathcal{T}_{\mathbf{A}_{(\alpha,\beta)}}$ and $|D \cap U| = |D \cap V| = \kappa$ as required, so we assume that case 2 holds at stage (α,β). Then $C_{(\alpha,\beta)}^{\varepsilon(\alpha,\beta)} \supseteq B_{(\alpha,\beta)}^{\varepsilon(\alpha,\beta)} \backslash (K_\alpha \cup \{\beta\})$, so $U \supseteq V \backslash (K_\alpha \cup \{\beta\})$, and from $|(D \cap V) \backslash K_\alpha| = \kappa$ then follows $|D \cap U| = \kappa$, as required.

Thus each family $\mathbf{A}_i$ is κ-independent and each $D \in \mathcal{D}$ is κ-dense in $\langle \kappa, \mathcal{T}_{\mathbf{A}_i} \rangle$, so $\mathbf{C} = \{\mathcal{C}_i \colon i \in I\}$ is κ-independent and each $D \in \mathcal{D}$ is κ-dense in $\langle \kappa, \mathcal{T}_{\mathbf{C}} \rangle$.

(a) having been proved, it remains to verify (b) and (c). Each nonempty $U \in \mathcal{T}_{\mathbf{C}}$ satisfies $|U| = \kappa$, so it suffices to show that the sets E hypothesized in (b) and (c) are closed and discrete.

We begin with this claim:

($*$) *If $E = K_\alpha$ has the property that for no $\beta < \kappa$ does case 1 arise at stage $i = (\alpha,\beta)$, then E is closed and discrete in $\mathcal{T}_{\mathbf{C}}$.*

Indeed for each β with $i = (\alpha,\beta)$ we have $\beta \in C_i^1 \in \mathcal{C}_i \subseteq \mathcal{T}_{\mathbf{C}}$ and

$$C_i^1 \cap K_\alpha = \left\{ \begin{array}{ll} \emptyset & \text{if } \beta \notin K_\alpha \\ \{\beta\} & \text{if } \beta \in K_\alpha \end{array} \right\},$$

so K_α is closed and discrete in $\mathcal{T}_{\mathbf{C}}$.

(b) is then immediate, since for $D \in \mathcal{D}$ and $\emptyset \neq U \in \mathcal{T}_{\mathbf{A}_i}$ always $|D \cap U| = \kappa$, so $|(D \cap U) \backslash K_\alpha| < \kappa$ cannot occur if $K_\alpha \in [\kappa]^{<\kappa}$.

Next we claim:

($**$) *If $\alpha < 2^\kappa$ and case 1 arises at stage $i = (\alpha,\beta)$ for some $\beta < \kappa$, then case 1 arises for all $i' = (\alpha,\beta')$ with $\beta' < \kappa$.*

To see this, let $|(D \cap U) \backslash K_\alpha| < \kappa$ with $D \in \mathcal{D}$ and, with

$$U = \bigcap_{j \in \mathrm{dom}(\varepsilon)} X_j \in \mathcal{T}_{\mathbf{A}_i} \quad \text{with } X_j = C_j^{\varepsilon(j)} \quad \text{if } j < i,$$
$$X_j = B_j^{\varepsilon(j)} \quad \text{if } j \geqslant i, \tag{$\dagger$}$$

define $V = \bigcap_{j \in \mathrm{dom}(\varepsilon)} Y_j \in \mathcal{T}_{\mathbf{A}_{i'}}$ by

$$Y_j = C_j^{\varepsilon(j)} \quad \text{if } j < i',$$
$$Y_j = B_j^{\varepsilon(j)} \quad \text{if } j = (\alpha, \beta_j) \geqslant i'.$$

Since $X_j \triangle Y_j \subseteq K_\alpha \cup \{\beta, \beta_j\}$ for all $j \in \mathrm{dom}(\varepsilon)$, there is a finite set $F \subseteq \kappa$ such that $U \triangle V \subseteq K_\alpha \cup F$. From $|(D \cap U) \backslash K_\alpha| < \kappa$ then follows $|(D \cap V) \backslash K_\alpha| < \kappa$, as asserted.

From ($*$) and ($**$) together follows:

(∗∗∗) *for every basic set* $U \in \mathcal{T}_{\mathbf{A}_i}$ *there is an open set* $V \in \mathcal{T}_{\mathbf{C}}$ *such that* $U \triangle V$ *is closed and discrete* (*hence nowhere dense*) *in* $\langle \kappa, \mathcal{T}_{\mathbf{C}} \rangle$.

To see this, let $U = \bigcap_{j \in \mathrm{dom}(\varepsilon)} X_j$ with $\varepsilon \in Fn(I, \lambda)$ as in (†), and note for each $j \in \mathrm{dom}(\varepsilon)$ with $i \leqslant j$ that if case 1 arises at stage $j = (\alpha_j, \beta_j)$ then $X_j = B_j^{\varepsilon(j)} = C_j^{\varepsilon(j)}$, while if case 2 arises then K_{α_j} is closed and discrete in $\mathcal{T}_{\mathbf{C}}$, so $K_{\alpha_j} \cup \{\beta_j\}$ is closed and discrete in $\mathcal{T}_{\mathbf{C}}$. Thus again with $V := \bigcap_{j \in \mathrm{dom}(\varepsilon)} C_j^{\varepsilon(j)}$ we have $V \in \mathcal{T}_{\mathbf{C}}$ and $U \triangle V$ is the union of finitely many sets each closed and discrete in $\mathcal{T}_{\mathbf{C}}$.

Finally we claim:

(∗∗∗∗) *If* $E = K_\alpha$ *has the property that case 1 arises at stage* $i = (\alpha, \beta)$ *for some* $\beta < \kappa$, *then there are* $D \in \mathcal{D}$ *and a nonempty set* $W \in \mathcal{T}_{\mathbf{C}}$ *such that* $E \supseteq D \cap W$.

Indeed, the set $N := (D \cap U) \backslash K_\alpha$ is closed, discrete and nowhere dense in $\mathcal{T}_{\mathbf{C}}$ by (b), and from (∗∗∗) there is $V \in \mathcal{T}_{\mathbf{C}}$ such that $M := U \triangle V$ is also closed, discrete and nowhere dense in $\mathcal{T}_{\mathbf{C}}$. Now $U \supseteq (D \cap U) \backslash N \supseteq (D \cap V) \backslash (N \cup M)$, so $E \supseteq D \cap W$ with $W := V \backslash (N \cup M) \neq \emptyset$, $W \in \mathcal{T}_{\mathbf{C}}$ as required.

Condition (c) is now clear, since for $E = K_\alpha$ as there hypothesized there is (by (∗∗∗∗)) no $\beta < \kappa$ such that case 1 arises at stage $i = (\alpha, \beta)$, so (∗) applies. □

Remarks 2.2. (a) It is clear that if the family $\mathcal{D}$ hypothesized in Theorem 2.1 is pairwise disjoint, then the space $\langle \kappa, \mathcal{T}_{\mathbf{C}} \rangle$ remains $|\mathcal{D}|$-resolvable. Under additional cardinality hypotheses consistent with ZFC, the space $\langle \kappa, \mathcal{T}_{\mathbf{C}} \rangle$ fails to enjoy certain resolvability properties. Details appear naturally in (the proof of) Theorem 3.1(c) below.

(b) The topology $\mathcal{T}_{\mathbf{C}}$ defined in the proof of Theorem 2.1 in terms of **B** and $\mathcal{D}$ has an additional property not needed explicitly later on (and hence not established in detail). For each set $E \subseteq \kappa$, these conditions are equivalent: (a) E is nowhere dense in $(\kappa, \mathcal{T}_{\mathbf{C}})$; (b) E is closed and discrete in $(\kappa, \mathcal{T}_{\mathbf{C}})$; and (c) $\mathrm{int}_{(D, \mathcal{T}_{\mathbf{C}})}(D \cap E) = \emptyset$ for each $D \in \mathcal{D}$.

Following Hewitt [19], we say that a space X is *maximally irresolvable* (MI, in [19]) if each dense subset of X is open. [More recently, many authors have chosen to refer to the topology on such a space as *submaximal*.] And, X is *hereditarily irresolvable* if no nonempty subspace of X is resolvable. A straightforward argument already noted in [19, Theorem 23] shows that every maximally irresolvable space is hereditarily irresolvable. The following noteworthy special case of Theorem 2.1, which for emphasis we state in slightly redundant form, closely parallels the results given in [24, 4.1]. Compare also our own earlier result [6, 5.4]: for $\kappa \geqslant \omega$ there is a dense, hereditarily irresolvable set $X \subseteq \{0, 1\}^{2^\kappa}$ such that $|X| = d(X) = \mathrm{nwd}(X) = \kappa$.

Theorem 2.3. *Let* $2 \leqslant \lambda \leqslant \kappa$ *and* $\kappa \geqslant \omega$. *There is a dense, hereditarily irresolvable, maximally irresolvable subspace* X *of* $(D(\lambda))^{2^\kappa}$ *such that* $|X| = d(X) = \Delta(X) = \kappa$ *and* $S(X) = \lambda^+$.

Proof. With $(\kappa, \mathcal{T}_{\mathbf{B}})$ as hypothesized in Theorem 2.1 let **C** be as given there with $\mathcal{D} = \{\kappa\}$, and set $X = e_{\mathbf{C}}[\kappa] \subseteq (D(\lambda))^{2^\kappa}$ with $e_{\mathbf{C}}$ defined as in 1.4. Clearly if A is dense in $X = (\kappa, \mathcal{T}_{\mathbf{C}})$ then by condition (c) of Theorem 2.1 the set $E := X \backslash A$ is closed and nowhere dense in X, and the required statements are immediate. □

3. Combinatorics, applied

We are able now to give our (consistent) negative response to Problem 1.1 from [24]: Is an extraresolvable Tychonoff space necessarily maximally resolvable? (The question specifically posed in [24] is whether an extraresolvable Tychonoff space X with $\Delta(X) \geqslant \omega^+$ must be ω^+-resolvable.) For use in Corollary 3.6, we state the theorem in slightly broader generality than is necessary to achieve that specific goal.

Theorem 3.1. *Let* λ, τ *and* κ *be cardinals such that* $2 \leqslant \lambda \leqslant \tau \leqslant \kappa$, *with* $\tau \geqslant \omega$ *and* $2^{<\tau} = \tau$. *Then there is a point-separating* κ*-independent family* **E** *of* λ*-partitions of* κ *such that the space* $\langle \kappa, \mathcal{T}_{\mathbf{E}} \rangle$

(a) *is* τ*-resolvable*;

(b) *has a family $\mathcal{E}$ of dense subsets, with $|\mathcal{E}| = 2^\tau$, such that if E_0, E_1 are distinct elements of $\mathcal{E}$ then $E_0 \cap E_1$ is nowhere dense*; *and*

(c) *is not τ'-resolvable, if τ' is a cardinal such that $\tau < \mathrm{cf}(\tau')$.*

Proof. The definition of $\mathbf{E}$ requires three constituent components, constructed independently and separately; we assemble these together in the paragraph below beginning "Finally".

First let $\mathbf{B}$ be a κ-independent family of λ-partitions of κ with the properties given by Theorem 1.6. Let $\mathcal{D}$ be a partition of κ into $\mathcal{T}_\mathbf{B}$-dense subsets, with $|\mathcal{D}| = \tau \leqslant \kappa$. Since $\mathrm{nwd}\langle\kappa, \mathcal{T}_\mathbf{B}\rangle = \kappa$, each $D \in \mathcal{D}$ is κ-dense in $(\kappa, \mathcal{T}_\mathbf{B})$; so Theorem 2.1 applies to give a κ-independent family $\mathbf{C} = \{\mathcal{C}_\alpha\colon \alpha < 2^\kappa\}$ of λ-partitions with the properties listed there.

Secondly, let $\mathbf{T} = (\mathbf{T}, \leqslant) = \bigcup\{\mathbf{T}_\xi\colon \xi < \tau\}$ be the (rooted) ever-branching binary tree of height τ. Here for $\xi < \tau$ we have written

$$\mathbf{T}_\xi = \bigl\{t \in \mathbf{T}\colon \{s \in \mathbf{T}\colon s < t\} \text{ is order-isomorphic to the ordinal } \xi\bigr\}.$$

Using $|\mathbf{T}| = |{}^{<\tau}2| = 2^{<\tau} = \tau = |\mathcal{D}|$, we index $\mathcal{D}$ by (the nodes of) $\mathbf{T}$; that is, we write $\mathcal{D} = \{D_t\colon t \in \mathbf{T}\}$. For $S \subseteq \tau$ we set $X(S) := \bigcup\{D_t\colon t \in \mathbf{T}_\xi,\ \xi \in S\} \subseteq \kappa$.

Thirdly, let $\mathbf{I} = \{\mathcal{I}_\gamma\colon \gamma < 2^\tau\}$ be a small-set-separating τ-independent family of λ-partitions of τ; here $\mathcal{I}_\gamma = \{I_\gamma^\eta\colon \eta < \lambda\}$. For $\gamma < 2^\tau$ we set $\mathcal{D}_\gamma := \{X(I_\gamma^\eta)\colon \eta < \lambda\}$, and $\mathbf{D} := \{\mathcal{D}_\gamma\colon \gamma < 2^\tau\}$. Then $\mathbf{D}$ is a family of λ-partitions of κ (which for each $\xi < \tau$ separates no two points of $X(\{\xi\})$.

Finally we set $\mathbf{E} = \mathbf{C} \cup \mathbf{D}$. Given a $\mathcal{T}_\mathbf{E}$-basic set $\mathbf{C}(\varepsilon) \cap \mathbf{D}(\delta)$ (with $\varepsilon \in Fn(2^\kappa, \lambda)$, $\delta \in Fn(2^\tau, \lambda)$), choose $\xi \in \bigcap_{\gamma \in \mathrm{dom}(\delta)} I_\gamma^{\delta(\gamma)} \subseteq \tau$ and then choose $t \in \mathbf{T}_\xi$. Then $D_t \subseteq \mathbf{D}(\delta)$, and since D_t is κ-dense in $\mathcal{T}_\mathbf{C}$ we have $|D_t \cap \mathbf{C}(\varepsilon) \cap \mathbf{D}(\delta)| = |D_t \cap \mathbf{C}(\varepsilon)| = \kappa$. This shows that $\mathbf{E}$, a family of λ-partitions of κ, is in fact a κ-independent family.

Now we show, much as in the proof of Theorem 3.6(c) in [7], that if $S \in [\tau]^{<\tau}$ then $X(S)$ is closed and nowhere dense in $(\kappa, \mathcal{T}_\mathbf{E})$. Indeed, given $x \in \kappa\backslash X(S)$, say with $x \in D_t$ with $t \in \mathbf{T}_\xi$, $\xi \notin S$, since $\mathbf{I}$ is small-set-separating there are $\gamma < 2^\tau$ and distinct $\eta, \eta' < \lambda$ such that $\eta \in I_\gamma^\eta$ and $S \subseteq I_\gamma^{\eta'}$; then $x \in D_t \subseteq X(I_\gamma^\eta) \in \mathcal{D}_\gamma \subseteq \mathcal{T}_\mathbf{D} \subseteq \mathcal{T}_\mathbf{E}$ and $X(I_\gamma^\eta) \cap X(S) = \emptyset$. Thus each $x \in \kappa\backslash X(S) = X(\tau\backslash S)$ has a $\mathcal{T}_\mathbf{E}$-neighborhood disjoint from $X(S)$, so $X(S)$ is closed. The argument of the previous paragraph shows that $\kappa\backslash X(S)$ is dense in $(\kappa, \mathcal{T}_\mathbf{E})$: each $\mathcal{T}_\mathbf{E}$-basic open set $\mathbf{C}(\varepsilon) \cap \mathbf{D}(\delta)$ has $|\mathbf{D}(\delta)| = \kappa$, so there is $\xi \in \mathbf{D}(\delta)\backslash S$ and again any $t \in \mathbf{T}_\xi$ satisfies $D_t \subseteq \kappa\backslash X(S) = X(\tau\backslash S)$, $D_t \subseteq \mathbf{D}(\delta)$, and $|D_t \cap \mathbf{C}(\varepsilon)| = \kappa$.

Now the properties (a), (b) and (c) of the space $(\kappa, \mathcal{T}_\mathbf{E})$ can be readily verified.

(a) Each $f \in {}^\tau\{0, 1\}$ determines a path $b_f \subseteq \mathbf{T}$. For each such f and $\xi < \tau$ there is a (unique) $t = t(f, \xi) \in \mathbf{T}$ such that $\mathbf{T}_\xi \cap b_f = \{t\}$. The preceding argument shows that the set $H(f) := \bigcup\{D_{t(f,\xi)}\colon \xi < \tau\}$ meets each $\mathcal{T}_\mathbf{E}$-basic set, i.e., each $H(f)$ is dense in $\langle\kappa, \mathcal{T}_\mathbf{E}\rangle$. Now for $\xi < \tau$ define $f_\xi \in {}^\tau\{0, 1\}$ by

$$f_\xi(\eta) = \begin{Bmatrix} 0 & \text{if } \eta < \xi \\ 1 & \text{if } \eta \geqslant \xi \end{Bmatrix},$$

and for $\xi < \tau$ set $H(\xi) := H(f_{\xi+1})\backslash H(f_\xi) = H(f_{\xi+1}) \cap X((\xi, \tau))$. The sets $H(\xi)$ are pairwise disjoint, and each is the intersection of a $\mathcal{T}_\mathbf{E}$-dense set with a $\mathcal{T}_\mathbf{E}$-dense open set, hence is $\mathcal{T}_\mathbf{E}$-dense.

(b) Each set $H(f)$ ($f \in 2^\tau$) is $\mathcal{T}_\mathbf{E}$-dense, and if $f, g \in 2^\tau$ first differ at $\xi < \tau$ then $H(f) \cap H(g)$ is a subset of the $\mathcal{T}_\mathbf{E}$-nowhere dense set $X([0, \xi))$.

(c) It suffices to show that even $\langle\kappa, \mathcal{T}_\mathbf{C}\rangle$ is not τ'-resolvable. Recall first from [23, p. 107] or [9, 3.27] that $S((D(\lambda))^{2^\kappa}) = \lambda^+$, i.e., the space $(D(\lambda))^{2^\kappa}$ admits no family of λ^+-many pairwise disjoint nonempty open subsets. The same inequality then holds for each set D dense in $(D(\lambda))^{2^\kappa}$, in particular for each $D_t \in \mathcal{D}$. Suppose now that there is a family $\mathcal{E}$ of pairwise disjoint dense subsets of $\langle\kappa, \mathcal{T}_\mathbf{C}\rangle$ with $|\mathcal{E}| = \tau'$. If for each $E \in \mathcal{E}$ there is $t(E) \in \mathbf{T}$ such that $\mathrm{int}_{(\mathbf{D}_{t(E)}, \mathcal{T}_\mathbf{C})}(D_{t(E)} \cap E) \neq \emptyset$, then since $\mathrm{cf}(\tau') > \tau = |\mathbf{T}|$ there is (fixed) $t \in \mathbf{T}$ such that $\mathrm{int}_{(D_t, \mathcal{T}_\mathbf{C})}(D_t \cap E) \neq \emptyset$ for τ'-many $E \in \mathcal{E}$. This contradiction shows that there is $E \in \mathcal{E}$ such that $\mathrm{int}_{(D, \mathcal{T}_\mathbf{C})}(D \cap E) = \emptyset$ for all $D \in \mathcal{D}$, so E is nowhere dense in $\langle\kappa, \mathcal{T}_\mathbf{C}\rangle$ by Theorem 2.1(c). □

Remarks 3.2. (a) The argument in the verification of Theorem 3.1(c) showing that there is $E \in \mathcal{E}$ such that $\mathrm{int}_{(D, \mathcal{T}_\mathbf{C})} D \cap E = \emptyset$ for all $D \in \mathcal{D}$ is as given in [7]. Indeed, as shown in [21], that relation holds for every $E \in \mathcal{E}$ with fewer than κ-many exceptions.

(b) The reader familiar with our earlier works [7,21] will recognize the passage in Theorem 3.1 from the family **C** of Theorem 2.2 to **E** via **D** as an instance of the **KID**-expansion process. Here **I** and **D** are as defined, and in the notation of those works the family **K** (not needed here) is the set $\emptyset \subseteq \kappa$ repeated 2^κ-many times.

We continue with consequences of Theorem 3.1. Only in 3.6 do we assume a hypothesis which fails in some models of ZFC. (The relation $\tau = 2^{<\tau}$ is satisfied in every model of ZFC by many cardinals τ, indeed by $\tau = \omega$ and by every strong limit cardinal.) We note in passing that the existence of cardinals τ for which there is σ such that $\tau = \sigma^+ = 2^\sigma$ cannot be established in ZFC; see in this connection [17].

In Theorem 4.5 of [24] it is established in ZFC that for every $\tau \geqslant \omega$ there is a Tychonoff space which is τ-resolvable but not τ^+-resolvable (and with some additional pre-assigned topological properties). Our contribution in this direction takes the following form.

Corollary 3.3. *Let $\tau \geqslant \omega$ satisfy $2^{<\tau} = \tau$. Then for every cardinal $\kappa \geqslant \tau^+$ there is a Tychonoff space X with $|X| = \kappa$ such that X is τ-resolvable and not τ^+-resolvable; for fixed $\lambda \in [\omega, \tau]$ one may arrange also $S(X) = \lambda^+$.*

Corollary 3.4. *For every uncountable cardinal κ there is a Tychonoff space X such that $|X| = \kappa$ and X is ω-resolvable but not maximally resolvable. For fixed $\lambda \in [\omega, \kappa)$ one may arrange in addition that $S(X) = \lambda^+$.*

Corollary 3.5. *For cardinals λ, τ and κ such that $2 \leqslant \lambda \leqslant \tau \leqslant \kappa$ with $\tau \geqslant \omega$, $2^{<\tau} = \tau$ and $2^\tau > \kappa$, there is a point-separating κ-independent family $\mathbf{E}$ of λ-partitions of κ such that the associated space $X = \langle \kappa, \mathcal{T}_\mathbf{E} \rangle$ is τ-resolvable, is extraresolvable, and is τ'-resolvable for no cardinal τ' such that $\tau < \mathrm{cf}(\tau')$.*

Proof. Indeed, the family $\mathcal{E}$ of Theorem 3.1 is now an "extraresolvable family" of cardinality $2^\tau > \kappa = \Delta(\kappa, \mathcal{T}_\mathbf{E})$. □

We say that GCH *first fails at* τ if $2^\mu = \mu^+$ for all $\mu < \tau$, and $2^\tau > \tau^+$.

Corollary 3.6. *Suppose that* GCH *first fails at τ. Then for $2 \leqslant \lambda \leqslant \tau$ there is a point-separating, τ^+-independent family* $\mathbf{E}$ *of λ-partitions of τ^+ such that the Tychonoff space $X = \langle \tau^+, \mathcal{T}_\mathbf{E} \rangle$ is τ-resolvable, extraresolvable and not maximally resolvable.*

Proof. This again is immediate from Theorem 3.1, taking $\tau' = \kappa = \tau^+$. □

4. Resolvability in $(D(\lambda))^I$

Discussion 4.1. In most of the preceding results we have dealt with κ-independent families of λ-partitions of κ, with $2 \leqslant \lambda \leqslant \kappa \geqslant \omega$, but in Corollary 3.4 the stronger hypothesis $\lambda < \kappa$ appeared. The question then arises naturally whether for $\omega \leqslant \kappa$ there is a (maximal) κ-independent family $\mathbf{B}$ of κ-partitions of κ such that $S(\kappa, \mathcal{T}_\mathbf{B}) = \kappa^+$ and $(\kappa, \mathcal{T}_\mathbf{B})$ is ω-resolvable but not maximally resolvable. To the best of our knowledge, this question was first posed explicitly in [20]. We can now respond.

The following theorem is transparent in case $2 \leqslant \lambda \leqslant \omega$, so we assume $\lambda > \omega$.

Theorem 4.2. *Let $\lambda \geqslant \omega$ and let X be dense in a space of the form $(D(\lambda))^I$. If X is ω-resolvable, then X is λ-resolvable.*

Proof. [In passing we make two remarks. (a) Obviously the conclusion can hold only if $\Delta(X) \geqslant \lambda$. It is easily seen that this condition follows from the density hypothesis, so it need not appear as a hypothesis. (b) If $|I| \leqslant \lambda$ then $w(X, \mathcal{T}_\mathbf{B}) = |I| \cdot \lambda = \lambda \leqslant \Delta(X, \mathcal{T}_\mathbf{B})$ and maximal resolvability of the space $(X, \mathcal{T}_\mathbf{B})$ is immediate from Ceder's theorem [2] (an application of the disjoint refinement lemma; the applicable argument does not require the auxiliary ω-resolvability hypothesis). As it happens, the argument to follow is valid for arbitrary infinite I; we take I well-ordered, with $\omega \subseteq I$.]

Let $\{D_n \colon n < \omega\}$ witness the ω-resolvability of X.

For $n < \omega \subseteq I$ and $\eta < \lambda$ set

$$E(n,\eta) := \{x \in X\colon 0 \leqslant k < n \Longrightarrow x_k \neq \eta = x_n\},$$

and define $E_\eta := \bigcup_{n<\omega}(E(n,\eta) \cap D_n)$ We will show that $\{E_\eta\colon \eta < \lambda\}$ is a family of (λ-many) pairwise disjoint dense subsets of E.

To see that $E_\eta \cap E_{\eta'} \neq \emptyset$ only when $\eta = \eta' < \lambda$, suppose there are x, n and n' such that $x \in (E(n,\eta) \cap D_n) \cap (E(n',\eta') \cap D_{n'})$. Then $n = n'$ (since otherwise $D_n \cap D_{n'} = \emptyset$), so $\eta = x_n = x_{n'} = \eta'$.

To see that each E_η is dense in X it suffices to show for each $\varepsilon \in Fn(I,\lambda)$ that some set $E(n,\eta)$ meets $\mathbf{B}(\varepsilon) := \bigcap_{i\in \mathrm{dom}(\varepsilon)}\{x\colon x_i = \varepsilon(i)\}$ (for then from the density of D_n will follow $\mathbf{B}(\varepsilon) \cap E_\eta \supseteq \mathbf{B}(\varepsilon) \cap E(n,\eta) \cap D_n \neq \emptyset$). Given such ε, let $m = \min\{n < \omega\colon m \notin \mathrm{dom}(\varepsilon)\}$. We assume without loss of generality, shrinking $\mathbf{B}(\varepsilon)$ if necessary by augmenting $\mathrm{dom}(\varepsilon)$, that $m > 0$ and $m = \{0, 1, \ldots, m-1\} \subseteq \mathrm{dom}(\varepsilon)$ (and $\mathrm{dom}(\varepsilon) \cap [m,\omega) = \emptyset$). We consider two cases.

Case 1. There is $n < m$ such that $\varepsilon(n) = \eta$. Choosing n minimal with that property we have $\emptyset \neq \mathbf{B}(\varepsilon) \subseteq E(n,\eta)$.

Case 2. Case 1 fails. Then with $\delta := \varepsilon \cup \{(m,\eta)\}$ we have $\emptyset \neq \mathbf{B}(\delta) \subseteq \mathbf{B}(\varepsilon) \cap E(m,\eta)$. □

Corollary 4.3. *Let $\omega < \lambda \leqslant \kappa$ and let $\mathbf{B} = \{\mathcal{B}_i\colon i \in I\}$ be a point-separating* 1*-independent family of λ-partitions of κ such that the space $X = \langle\kappa, \mathcal{T}_{\mathbf{B}}\rangle$ is ω-resolvable. Then*

(a) *X is maximally resolvable*; *and*
(b) *if $\mathbf{B}$ is λ-independent then $\mathbf{B}$ is not maximal among λ-independent families of λ-partitions of κ.*

Remarks 4.4. (a) As indicated earlier, the question whether every ω-resolvable (Tychonoff) space is maximally resolvable, solved recently in ZFC in [24] and in 3.1–3.4 above, dates back to 1967 [3]. Theorem 4.2 helps to explain why the question proved so difficult: Among spaces which lie in what is perhaps the most natural and fruitful and the most fully investigated setting in the study of resolvability counterexamples, the question has a positive answer.

(b) Theorem 4.2 strikes us as unexpected. It shows that in quite general circumstances a small (that is, countable) family of pairwise disjoint dense sets can be parlayed into a family of large cardinality. Juxtaposed against Theorem 3.1, it defines a "breaking point" or stage of separation in the degree of resolvability that is assured: If λ is a strong limit cardinal (the case $\lambda = \omega$ being permitted) or has the form $\lambda = \sigma^+ = 2^\sigma$, and if $\kappa > \lambda$ with X dense in $(D(\lambda))^{2^\kappa}$, $|X| = \kappa$, then X, if ω-resolvable, is necessarily λ-resolvable; but according to Theorem 3.1 such X exist which are not λ^+-resolvable.

(c) Although Theorem 4.2 indicates that in appropriate instances ω-resolvability implies λ-resolvability, it should be noted that one cannot manufacture a large resolvable family of sets completely from scratch. That is, a dense subspace of a space of the form $(D(\lambda))^I$ need not be ω-resolvable, nor even resolvable; see Theorem 2.3 above or our earlier result [6, 5.4].

So far as we can discern, our methods of proof fall short of providing a response to the following question, which strikes us therefore as an attractive point of departure for further study.

Question 4.5. Let $X = \langle X, \mathcal{T}\rangle$ be an ω-resolvable Tychonoff space in which each nonempty $U \in \mathcal{T}$ satisfies $S(U) = (\Delta(X))^+$. Is X necessarily maximally resolvable?

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