Available online at www.sciencedirect.com
SCIENCE DIRECT®
Topology and its Applications 153 (2006) 2192–2198
Topology and its Applications
www.elsevier.com/locate/topol

The embeddability ordering of topological spaces ☆

W.W. Comfort *,1, W.D. Gillam

Wesleyan University, Middletown, CN, USA

Received 10 August 2003; accepted 16 February 2004

Respectfully dedicated to Alexander "Shura" Arhangel'skiĭ on the occasion of his 65 th birthday

Abstract

For $\mathcal{K}$ a set of topological spaces and $X, Y \in \mathcal{K}$, the notation $X \subseteq_h Y$ means that X embeds homeomorphically into Y; and $X \sim Y$ means $X \subseteq_h Y \subseteq_h X$. With $\widetilde{X} := \{Y \in \mathcal{K}\colon\ X \sim Y\}$, the equivalence relation $\sim$ on $\mathcal{K}$ induces a partial order $\leqslant_h$ well-defined on $\mathcal{K}/\sim$ as follows: $\widetilde{X} \leqslant_h \widetilde{Y}$ if $X \subseteq_h Y$.

For posets $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$, the notation $(P, \leqslant_P) \hookrightarrow (Q, \leqslant_Q)$ means: there is an injection $h : P \to Q$ such that $p_0 \leqslant_P p_1$ in P if and only if $h(p_0) \leqslant_Q h(p_1)$ in Q. For κ an infinite cardinal, a poset $(Q, \leqslant_Q)$ is a κ-*universal* poset if every poset $(P, \leqslant_P)$ with $|P| \leqslant \kappa$ satisfies $(P, \leqslant_P) \hookrightarrow (Q, \leqslant_Q)$.

The authors prove two theorems which improve and extend results from the extensive relevant literature.

Theorem 2.2. *There is a zero-dimensional Hausdorff space S with $|S| = \kappa$ such that $(\mathcal{P}(S)/\sim, \leqslant_h)$ is a κ-universal poset.*

Theorem 3.1. *There are a compact, connected Hausdorff space S and a set $\mathcal{K}$ of (2^κ-many) compact, connected subspaces of S such that* (a) *the posets $(\mathcal{P}(\kappa), \subseteq)$ and $(\mathcal{K}/\sim, \leqslant_h)$ are isomorphic*; *and* (b) $(\mathcal{K}/\sim, \leqslant_h)$ *is a κ-universal poset. Further, one may arrange $|S| = w(S) = |X| = w(X) = \aleph_\kappa \cdot \mathfrak{c}$ for each $X \in \mathcal{K}$.*

☆ Some of the results in this paper appear in the Master's Thesis of the second-listed co-author [W.D. Gillam, The embeddability ordering of topological spaces, Master of Arts thesis, Wesleyan University, Middletown, CN, 2003 [6]], written under the supervision of the first-listed co-author.

* Corresponding author.

E-mail addresses: wcomfort@wesleyan.edu (W.W. Comfort), wgillam@wesleyan.edu (W.D. Gillam).

1 Presented by this co-author at the Conference honoring A.V. Arhangel'skiĭ and his work (June 2003, Brooklyn College).

0166-8641/$ – see front matter © 2005 Elsevier B.V. All rights reserved.
doi:10.1016/j.topol.2004.02.023

MSC: primary 54H10; secondary 06A06

Keywords: Homeomorphic spaces; Partially ordered set; Poset embedding; Universal poset

1. Introduction

1.1. Notation

The symbols κ, λ and μ denote cardinal numbers, and ξ and η are ordinals. For S a set we denote by $\mathcal{P}(S)$ the power set of S, and when κ is a cardinal we write $[S]^\kappa = \{A \in \mathcal{P}(S)\colon |A| = \kappa\}$. When S is a space the elements of $\mathcal{P}(S)$ are themselves spaces (with the inherited topology).

For $\mathcal{K}$ a set of topological spaces and $X, Y \in \mathcal{K}$, the notation $X \subseteq_h Y$ means that X is homeomorphic to a subspace of Y; when $X \subseteq_h Y \subseteq_h X$ we write $X \sim Y$, and we set $\widetilde{X} := \{Y \in \mathcal{K}\colon X \sim Y\}$. The set $\mathcal{K}/{\sim}$ of equivalence classes is partially ordered by the relation: $\widetilde{X} \leqslant_h \widetilde{Y}$ if $X \subseteq_h Y$; the resulting poset is denoted $\widetilde{\mathcal{K}} := (\mathcal{K}/{\sim}, \leqslant_h)$. In particular, for a space S we have $\widetilde{\mathcal{P}(S)} := (\mathcal{P}(S)/{\sim}, \leqslant_h)$.

For κ an infinite cardinal, a poset $(Q, \leqslant_Q)$ is a κ-*universal* poset if every poset $(P, \leqslant_P)$ with $|P| \leqslant \kappa$ satisfies $(P, \leqslant_P) \hookrightarrow (Q, \leqslant_Q)$. We remark for emphasis that according to our usage a κ-universal poset $(Q, \leqslant_Q)$ need not satisfy $|Q| \leqslant \kappa$.

1.2. Background

The literature in this corner of mathematics normally addresses two types of questions, as follows:

(A) A space S is given, and one asks: which posets $(P, \leqslant_P)$ satisfy $(P, \leqslant_P) \hookrightarrow \widetilde{\mathcal{P}(S)}$?
(B) A poset $(P, \leqslant_P)$ is given, and one asks: Is there a space S such that $(P, \leqslant_P) \hookrightarrow \widetilde{\mathcal{P}(S)}$?

[For every $(P, \leqslant_P)$ the answer to (B) is "Yes", so attention is typically directed to more subtle questions, such as these two.]

(i) What is $\min\{|S|\colon (P, \leqslant_P) \hookrightarrow \widetilde{\mathcal{P}(S)}\}$?
(ii) Can S, and/or the elements of $P \hookrightarrow \mathcal{P}(S)$, be chosen with special pre-assigned topological properties—e.g., connected, compact, ... ?

1.3. Discussion

(a) In the spirit of (A), with $S = \mathbb{R}$, we cite a pleasing recent result of McCluskey et al. [14]: $(\mathcal{P}(\mathfrak{c}), \subseteq) \hookrightarrow \widetilde{\mathcal{P}(\mathbb{R})}$. It follows in particular, since $(\mathcal{P}(\mathfrak{c}), \subseteq)$ is easily seen to contain an antichain of cardinality $2^{\mathfrak{c}}$, that $\mathbb{R}$ contains a set of $2^{\mathfrak{c}}$-many subspaces, no one of which embeds homeomorphically into another; that specific statement dates back to 1926 (cf. Kuratowski [11]). The relation $(\mathcal{P}(\mathfrak{c}), \subseteq) \hookrightarrow \widetilde{\mathcal{P}(\mathbb{R})}$ does not tell the full story about $\widetilde{\mathcal{P}(\mathbb{R})}$, however, since $(\mathcal{P}(\mathfrak{c}), \subseteq)$ contains no well-ordered set of cardinality $\mathfrak{c}^+$ while $\widetilde{\mathcal{P}(\mathbb{R})}$ does contain such a set [12].

(b) Similarly in the spirit of (A), one of us has recently achieved a poset-theoretic characterization of the poset $\widetilde{\mathcal{P}(\mathbb{Q})}$ [7].

(c) In the direction of (B), McCluskey and McMaster [13] have shown that for $\kappa \geqslant \omega$ there is a space S with $|S| = \kappa$ such that $\widetilde{\mathcal{P}(S)}$ is a κ-universal poset. The spaces S constructed in [13] have the advantage that they contain large connected subspaces; but they are neither T_1-spaces nor regular, hence they are not Tychonoff spaces. In contrast, our argument in Theorem 2.2 below furnishes for each $\kappa \geqslant \omega$ a Tychonoff space $S = S_\kappa$ of cardinality κ such that $\widetilde{\mathcal{P}(S)}$ is a κ-universal poset.

(d) We note that Trňková [17] has established a strong version of the statement that the class of Tychonoff spaces contains antichains of arbitrarily large cardinality. Indeed, according to [17], there is a proper class $\mathcal{K}$ of paracompact Hausdorff spaces such that

(i) each X in $\mathcal{K}$ is *strongly rigid* in the sense that each continuous $f : X \to X$ satisfies either $f = \mathrm{id}_X$ or f is a constant function (i.e., $|f[X]| = 1$); and
(ii) the class $\mathcal{K}$ is *strongly bi-rigid* in the sense that for distinct X, Y in $\mathcal{K}$ each continuous $f : X \to Y$ satisfies $|f[X]| = 1$.

(e) Though it plays no role in this paper, we cannot resist citing this theorem of Schroeder–Bernstein type established in 1924 by Banach [1]: If spaces X and Y satisfy $X \sim Y$, then there are homeomorphic subsets A and B of X and Y, respectively, such that $X \backslash A$ and $Y \backslash B$ are also homeomorphic.

(f) It has been noted over the years by many workers that each poset $(P, \leqslant_P)$ embeds into the poset $(\mathcal{P}(P), \subseteq)$ (in symbols: $(P, \leqslant_P) \hookrightarrow (\mathcal{P}(P), \subseteq)$). Indeed the map $h : P \to \mathcal{P}(P)$ given by $h(x) = \{y \in P\colon\ y \leqslant_P x\}$ is an embedding as required. We use this fact frequently in what follows, without additional warning.

(g) It follows from the cited relation $(P, \leqslant_P) \hookrightarrow (\mathcal{P}(P), \subseteq)$ that for $\kappa \geqslant \omega$ the poset $(\mathcal{P}(\kappa), \subseteq)$ is a κ-universal poset (of cardinality 2^κ).

2. A small, zero-dimensional Tychonoff space S with $\widetilde{\mathcal{P}(S)}$ κ-universal

For $\omega \leqslant \alpha \leqslant \kappa$, a family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ is said to be *α-almost disjoint* if distinct elements A, B of $\mathcal{A}$ satisfy $|A \cap B| < \alpha$.

When κ is an infinite cardinal, the symbol $\beta(\kappa)$ denotes the Stone–Čech compactification of the (discrete) space κ. An ultrafilter $p \in \beta(\kappa)$ is *uniform* over κ if $p \subseteq [\kappa]^\kappa$; the set of ultrafilters uniform over κ is denoted $U(\kappa)$. When $A \subseteq \kappa$ we have $\beta(A) = \mathrm{cl}_{\beta(\kappa)} A$ and we write $U(A) := U(\kappa) \cap \beta(A)$. (According to this convention we have $U(A) = \emptyset$, if $|A| < \kappa$.) Given $A, B \subseteq \kappa$ and $f : A \to B \subseteq \beta(\kappa)$ we denote by $\overline{f}$ that continuous function $\overline{f} : \beta(A) \to \beta(B)$ such that $\overline{f}|A = f$.

Lemma 2.1. *Let $\omega \leqslant \kappa \leqslant \lambda$ and suppose there is a κ-almost disjoint family $\mathcal{A} \subseteq [\kappa]^\kappa$ such that $|\mathcal{A}| = \lambda$. Then there is a set $S \subseteq \beta(\kappa)$ such that*

(a) $|S| = \lambda$, *and*
(b) $\widetilde{\mathcal{P}(S)}$ *is a* κ*-universal poset.*

Proof. For $A \in \mathcal{A}$ we choose $p_A \in U(A)$ such that if $\{A, B\} \in [\mathcal{A}]^2$ there is no injection $f : A \to B \subseteq \beta(B)$ such that $\overline{f}(p_A) = p_B$. [The availability of such a set $\{p_A\colon A \in \mathcal{A}\}$ follows from well-known facts which are elucidated in detail in [2]. We content ourselves here with a brief outline. First, adopting terminology of Rudin [16] and Frolík [4,5] to the effect that the *type* $\tau(p)$ of an ultrafilter $p \in \beta(\kappa)$ is the set

$$\tau(p) = \bigl\{q \in \beta(\kappa)\colon \text{there is a permutation } f : \kappa \to \kappa \text{ such that } \overline{f}(p) = q\bigr\},$$

one has $|\tau(p)| \leqslant \kappa^\kappa = 2^\kappa$ (with equality easily shown). From $|U(\kappa)| = 2^{2^\kappa}$ (cf. [2, (7.8)]) it then follows that the set $T := \{\tau(p)\colon p \in U(\kappa)\}$ of "uniform types" satisfies $|T| = 2^{2^\kappa}$. It is known, further, that if $A \in [\kappa]^\kappa$ and $p \in U(\kappa)$ then $U(A) \cap \tau(p) \neq \emptyset$ (cf. [2]). It is enough, then, to choose an injection $i : \mathcal{A} \to T$ and then for $A \in \mathcal{A}$ to choose $p_A \in U(A)$ such that $\tau(p_A) \in i(A)$.] We claim that for $A \in \mathcal{A}$ the relation $\kappa \cup \{p_A\} \subseteq_h \kappa \cup \{p_B\colon B \in \mathcal{A} \setminus \{A\}\}$ fails. Indeed, suppose that $h : \kappa \cup \{p_A\} \to \kappa \cup \{p_B\colon B \in \mathcal{A} \setminus \{A\}\}$ is an embedding. Then $h(p_A) \notin \kappa$ (since p_A is not isolated in $\kappa \cup \{p_A\}$), so $h(p_A) = p_B$ for some $B \in \mathcal{A} \setminus \{A\}$. The set $\mathrm{cl}_{\beta(\kappa)} B$ is an open neighborhood in $\beta(\kappa)$ of p_B, and $\mathrm{cl}_{\beta(\kappa)} B \cap \{p_A\colon A \in \mathcal{A}\} = \{p_B\}$ since the family $\mathcal{A}$ is κ-almost disjoint and the ultrafilters p_A are uniform. Since $p_B \in \mathrm{cl}_{\beta(\kappa)}(h[\kappa] \cap B)$ and p_B is uniform, we have $|h[\kappa] \cap B| = \kappa$, so $h[\kappa] \cap B$ can be partitioned into two complementary sets U and V, each of cardinality κ, say with $U \in p_B$ and $V \notin p_B$. Then any permutation $h' : \kappa \to \kappa$ such that $h'|h^{-1}[U] = h|h^{-1}[U]$ and $h'[\kappa \backslash h^{-1}[U]] = \kappa \backslash U$ satisfies $\overline{h'}(p_A) = \bar{h}(p_A) = p_B$ and hence witnesses the relation $\tau(p_A) = \tau(p_B)$, a contradiction. The claim is established.

We define $S := \kappa \cup \{p_A\colon A \in \mathcal{A}\}$, and for $\mathcal{X} \subseteq \mathcal{A}$ we set $S_\mathcal{X} := \kappa \cup \{p_A\colon A \in \mathcal{X}\}$. Clearly for $\mathcal{X} \subseteq \mathcal{Y} \subseteq \mathcal{A}$ we have $S_\mathcal{X} \subseteq S_\mathcal{Y}$ and hence $\widetilde{S_\mathcal{X}} \leqslant_h \widetilde{S_\mathcal{Y}}$. If $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{A}$ and $\mathcal{X} \subseteq \mathcal{Y}$ fails (say $A \in \mathcal{X} \backslash \mathcal{Y}$) then $\mathcal{Y} \subseteq \kappa \cup \{p_B\colon B \in \mathcal{A} \setminus \{A\}\}$, so according to the claim established above, the relation $\kappa \cup \{p_A\} \subseteq_h \mathcal{S}_\mathcal{Y}$ fails and hence $\widetilde{S_\mathcal{X}} \leqslant_h \widetilde{S_\mathcal{Y}}$ fails. The map $\mathcal{X} \to S_\mathcal{X}$ thus establishes the relation $(\mathcal{P}(\mathcal{A}), \subseteq) \hookrightarrow \widetilde{\mathcal{P}(S)}$, and for every poset $(P, \leqslant_P)$ with $|P| \leqslant \lambda$ we have

$$(P, \leqslant_P) \hookrightarrow \bigl(\mathcal{P}(P), \subseteq\bigr) = \bigl(\mathcal{P}(\mathcal{A}), \subseteq\bigr) \hookrightarrow \widetilde{\mathcal{P}(S)},$$

as required. □

Theorem 2.2. *Let* $\kappa \geqslant \omega$. *There are zero-dimensional Tychonoff spaces* S_0 *and* S_1 *such that* $|S_0| = \kappa$, $|S_1| = \kappa^+$, $w(S_i) \leqslant 2^\kappa$, *and*

(0) $\widetilde{\mathcal{P}(S_0)}$ *is a* κ*-universal poset*; *and*
(1) $\widetilde{\mathcal{P}(S_1)}$ *is a* κ^+*-universal poset.*

If in addition $\kappa = 2^{<\kappa} := \sum_{\mu<\kappa} 2^\mu$, *then there is a zero-dimensional Tychonoff space* S_2 *such that* $|S_2| = 2^\kappa$, $w(S_i) = 2^\kappa$, *and*

(2) $\widetilde{\mathcal{P}(S_2)}$ *is a* 2^κ*-universal poset.*

Proof. In each case (i) $(0 \leqslant i \leqslant 2)$ we take $S_i = \kappa \cup \{p_A\colon\ A \in \mathcal{A}_i\} \subseteq \beta(\kappa)$ as in Lemma 2.1, with $|\mathcal{A}_0| = \lambda = \lambda_0 := \kappa$, $|\mathcal{A}_1| = \lambda = \lambda_1 := \kappa^+$, and $|\mathcal{A}_2| = \lambda = \lambda_2 := 2^\kappa$. That such a κ-almost disjoint family $\mathcal{A}_0 \subseteq \mathcal{P}(\kappa)$ exists is obvious; that such $\mathcal{A}_1$ exists is well known (see for example [18, p. 2] or [2, 12.6]); the existence of such $\mathcal{A}_2$ (when $\kappa = 2^{<\kappa}$) is a theorem of Tarski (see [2, (12.2)]). □

Remark 2.3. *En route* to their construction mentioned above of a connected, non-Tychonoff, countable space S such that $\widetilde{\mathcal{P}(S)}$ is an ω-universal poset, the authors of [13] demonstrate (by a direct, inductive argument) for $\kappa \geqslant \omega$ the existence of 2^{2^κ}-many distinct ultrafilter types over κ.

Discussion 2.4. The spaces S_0 of Theorem 2.2(0) for which $\widetilde{\mathcal{P}(S_0)}$ is a κ-universal poset satisfy $|\widetilde{\mathcal{P}(S_0)}| = 2^{|S_0|} = 2^\kappa$. The present authors do not know whether, given suitably restricted $\kappa \geqslant \omega$, there is a space S with $|S| < \kappa$ such that $\widetilde{\mathcal{P}(S)}$ is a κ-universal poset. (Naively, since $|\mathcal{P}(\log(\kappa))| \geqslant \kappa$, $|S| = \log(\kappa) < \kappa$ might occur—but we know of no such instance.) Specifically, we ask:

(A) Given $\kappa \geqslant \omega$, what is $\min\{|S|\colon\ \widetilde{\mathcal{P}(S)}$ is a κ-universal poset$\}$?

In this context, the following even more fundamental question arises.

(B) Given $\kappa \geqslant \omega$, what is the least cardinal of a κ-universal poset?

Let us denote that cardinal by the symbol $\mathfrak{m}(\kappa)$. It is known that $\mathfrak{m}(\omega) = \omega$; indeed, the witnessing countable poset S can be chosen in addition to be *ω-homogeneous* in the sense that every isomorphism between finite subposets extends to an automorphism of S. (More generally, for every Jónsson Class $\mathbb{K}$ and for $\omega \leqslant \alpha \leqslant \kappa = \kappa^{<\alpha}$ there is an α-homogeneous, α-universal $S \in \mathbb{K}$ such that $|S| = \kappa$; a κ-homogeneous, κ-universal $S \in \mathbb{K}$ with $|S| = \kappa$ exists if and only if $\kappa = \kappa^{<\kappa}$. See [15] or [2, §4] for detailed proofs in a broad context, and see [8, §6] for a treatment in modern language of the countable case.) In any case, the statement $\kappa \geqslant \omega \Rightarrow \mathfrak{m}(\kappa) = \kappa$ is not a theorem of ZFC: Arguing in ZFC, Kojman and Shelah [10] show that $\mathfrak{m}(\kappa) > \kappa$ for many singular cardinals κ, and for regular κ such that $\omega^+ < \kappa < \mathfrak{c}$. (The treatment in [10] focuses on linearly ordered sets rather than posets, but κ-universal structures of cardinality κ exist in those two classes for exactly the same κ.)

For classic studies which stimulated subsequent work on κ-universal posets, see Dushnik and Miller [3] and Johnston [9].

Remark 2.5. The referee has observed, in effect, that Question 2.4(A) invokes a spectrum of questions: Given $\kappa \geqslant \omega$ and classes $\mathbb{P}$ and $\mathbb{Q}$ of spaces, is there a space $S \in \mathbb{P}$ such that each poset $(P, \leqslant_P)$ with $|P| \leqslant \kappa$ admits an embedding $h : (P, \leqslant_P) \hookrightarrow \widetilde{\mathcal{P}(S)}$ such that each equivalence class $\widetilde{h(x)} \in \widetilde{\mathcal{P}(S)}$ contains a space in $\mathbb{Q}$? When such S exists, what is the smallest achievable cardinality?

3. A large, compact, connected space S with $\widetilde{\mathcal{P}(S)}$ κ-universal

Theorem 3.1. *Let $\kappa \geqslant \omega$. There are a compact, connected, Tychonoff space S and a family $\mathcal{K}$ of compact, connected subspaces of S such that the posets $(\mathcal{P}(\kappa), \subseteq)$ and $\widetilde{\mathcal{K}}$ are isomorphic. One may arrange that $|S| = w(S) = |X| = w(X) = \aleph_\kappa \cdot \mathfrak{c}$ for each $X \in \mathcal{K}$.*

Proof. Let $L := (\aleph_\kappa \times [0, 1), \text{lex}) \cup \{\aleph_\kappa\}$ be the one-point compactification of "the long ray of length $\aleph_\kappa$", and for each successor cardinal $\lambda = \aleph_{\xi+1}$ ($\xi < \kappa$) let $L(\lambda)$ denote the space L with the interval $I_\lambda := [0, 1]$ attached at the point $(\lambda, 0)$. (More precisely: $L(\lambda)$ is the disjoint union of L with I_λ, with the points $(\lambda, 0) \in L$ and $0 \in I_\lambda$ identified.)

We claim that for $\lambda = \aleph_{\xi+1}$ and $\mu = \aleph_{\eta+1}$ with $\{\xi, \eta\} \in [\kappa]^2$ the relation $L(\lambda) \subseteq_h L(\mu)$ fails. The point $(\lambda, 0)$ of $L(\lambda)$ has the property that

(i) the local weight of $L(\lambda)$ at $(\lambda, 0)$ satisfies $\chi(L(\lambda), (\lambda, 0)) = \lambda = \aleph_{\xi+1}$, and
(ii) the space $L(\lambda)\backslash\{(\lambda, 0)\}$ has three connected components.

No connected subspace of $L(\mu)$ contains a point with local weight λ whose deletion yields a space with three components, so there is no injective homeomorphism from $L(\lambda)$ into $L(\mu)$. The claim is established.

For $\mathcal{X} \subseteq \kappa$ we set $S_\mathcal{X} := L \cup \bigcup\{L(\lambda)\colon \lambda = \aleph_{\xi+1},\ \xi < \kappa,\ \xi \in \mathcal{X}\}$, with all "base points" $(0, 0) \in L$ and $(0, 0) \in L(\lambda)$ identified. (Thus, $L_\emptyset = L$.) $L(\lambda)$ is topologized so that each non-base point of L and each non-base point of $L(\lambda)$ has the same basic open neighborhood family as originally, while basic neighborhoods of the new base point $(0, 0)$ are given by restriction in finitely many rays; that is, a set U containing $(0, 0)$ is open if

(i) $U \cap L$ is open in the original topology of L;
(ii) for $\lambda = \aleph_{\xi+1}$, $\xi \in \mathcal{X}$, $U \cap L(\lambda)$ is open in the original topology of $L(\lambda)$; and
(iii) the inclusion $L(\lambda) \subseteq U$ fails for at most finitely many λ.

It is clear that each $S_\mathcal{X}$ is a compact, connected, Hausdorff space.

We define $\mathcal{K} := \{S_\mathcal{X}\colon \mathcal{X} \subseteq \kappa\}$. Clearly for $\mathcal{X} \subseteq \mathcal{Y} \subseteq \kappa$ we have $S_\mathcal{X} \subseteq S_\mathcal{Y}$ and hence $\widetilde{S_\mathcal{X}} \leqslant_h \widetilde{S_\mathcal{Y}}$. If $\mathcal{X}, \mathcal{Y} \subseteq \kappa$ and $\mathcal{X} \subseteq \mathcal{Y}$ fails (say $\xi \in \mathcal{X}\backslash\mathcal{Y}$) then according to the claim just established the relation $L(\aleph_{\xi+1}) \subseteq_h \mathcal{S}_\mathcal{Y}$ fails and hence $\widetilde{S_\mathcal{X}} \leqslant_h \widetilde{S_\mathcal{Y}}$ fails. The map $\mathcal{X} \to S_\mathcal{X}$ thus establishes a poset isomorphism from $(\mathcal{P}(\kappa), \subseteq)$ onto $\widetilde{\mathcal{K}}$, as required. □

Remark 3.2. From earlier observations it follows for S as in Theorem 3.1 that $\widetilde{\mathcal{P}(S)}$ is κ-universal. The principal point here is that for every poset $(P, \leqslant_P)$ with $|P| \leqslant \kappa$ the witnessing embedding $h : (P, \leqslant_P) \hookrightarrow \widetilde{\mathcal{K}} \subseteq \widetilde{\mathcal{P}(S)}$ can be achieved so that $h(x)$ has a compact and connected representative for each $x \in P$. For all we know, the cardinal number $|S| = \aleph_\kappa \cdot \mathfrak{c}$ in the proof just given may be larger than necessary. In parallel with the questions posed in Section 2, we ask:

Question 3.3. Let $\kappa \geqslant \omega$. What is the minimum cardinality of a compact, connected, Tychonoff space S such that every poset $(P, \leqslant_P)$ with $|P| \leqslant \kappa$ admits an embedding

$h:(P,\leqslant_P)\hookrightarrow\widetilde{\mathcal{P}(S)}$ with $h(x)$ containing a compact and connected representative for each $x\in P$?

Acknowledgement

On the matter of κ-universal structures (see Section 2.4) we acknowledge with thanks helpful discussions and correspondence with Rehana Patel, Katherine Thompson, and Carol Wood. We are grateful also for several useful expository suggestions (and corrections) offered by the referee.

References

[1] S. Banach, Un théorème sur les transformations biunivoque, Fund. Math. 6 (1924) 236–239.
[2] W.W. Comfort, S. Negrepontis, The Theory of Ultrafilters, Grundlehren Math. Wiss. Einzeldarstellungen, vol. 221, Springer, Berlin, 1974.
[3] B. Dushnik, E.W. Miller, Partially ordered sets, Amer. J. Math. 63 (1941) 600–610.
[4] Z. Frolík, Types of ultrafilters on countable sets, in: J. Novák (Ed.), General Topology and Its Relations to Modern Analysis and Algebra, II, Proc. Second Prague Topological Symposium, Academia Press, Prague, 1966, pp. 142–143.
[5] Z. Frolík, Sums of ultrafilters, Bull. Amer. Math. Soc. 73 (1967) 87–91.
[6] W.D. Gillam, The embeddability ordering of topological spaces, Master of Arts thesis, Wesleyan University, Middletown, CN, 2003.
[7] W.D. Gillam, Embeddability properties of countable metric spaces, Topology Appl. 148 (2005) 63–82.
[8] W. Hodges, A Shorter Model Theory, Cambridge University Press, Cambridge, 1997.
[9] J.B. Johnston, Universal infinite partially ordered sets, Proc. Amer. Math. Soc. 7 (1956) 507–514.
[10] M. Kojman, S. Shelah, Non-existence of universal orders in many cardinals, J. Symbolic Logic 57 (3) (1992) 857–891.
[11] C. Kuratowski, Sur la puissance de l'ensemble des "nombres de dimension" au sens de M. Fréchet, Fund. Math. 8 (1926) 201–208.
[12] C. Kuratowski, W. Sierpiński, Sur un problème de M. Fréchet concernant les dimensions des ensembles linéaries, Fund. Math. 8 (1926) 193–200.
[13] A.E. McCluskey, T.B.M. McMaster, Representing quasi-orders by embeddability ordering of families of topological spaces, Proc. Amer. Math. Soc. 127 (2000) 1275–1279.
[14] A.E. McCluskey, T.B.M. McMaster, W.S. Watson, Representing set-inclusion by embeddability (among the subspaces of the real line), Topology Appl. 96 (1999) 89–92.
[15] M. Morley, R. Vaught, Homogeneous universal models, Math. Scand. 11 (1962) 37–57.
[16] W. Rudin, Homogeneity problems in the theory of Čech compactifications, Duke Math. J. 23 (1956) 409–419, 633.
[17] V. Trňková, Full embeddings into the categories of boolean algebras, Comment. Math. Univ. Carolin. 27 (1986) 535–541.
[18] N. Williams, Combinatorial Set Theory, North-Holland, Amsterdam, 1977.