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Maximal independent families and a topological consequence ☆

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Abstract

For $\kappa \geqslant \omega$ and X a set, a family $\mathcal{A} \subseteq \mathcal{P}(X)$ is said to be κ-*independent* on X if $|\bigcap_{A\in\mathcal{F}} A^{f(A)}| \geqslant \kappa$ for each $\mathcal{F} \in [\mathcal{A}]^{<\omega}$ and $f \in \{-1,+1\}^{\mathcal{F}}$; here $A^{+1} = A$ and $A^{-1} = X\backslash A$.

Theorem 3.6. *For $\kappa \geqslant \omega$, some $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ with $|\mathcal{A}| = 2^\kappa$ is simultaneously maximal κ-independent and maximal ω-independent on κ. The family $\mathcal{A}$ may be chosen so that every two elements of κ are separated by 2^κ-many elements of $\mathcal{A}$.*

Corollary 5.4. *For $\kappa \geqslant \omega$ there is a dense subset D of $\{0,1\}^{2^\kappa}$ such that each nonempty open $U \subseteq D$ satisfies $|U| = d(U) = \kappa$ and no subset of D is resolvable. The set D may be chosen so that every two of its elements differ in 2^κ-many coordinates.*

Remarks. (a) Theorem 3.6 answers affirmatively a question of Eckertson [Topology Appl. 79 (1997) 1–11]. Two proofs are given here. (b) Parts of Corollary 5.4 have been obtained by other methods by Feng [Topology Appl. 105 (2000) 31–36] and (for $\kappa = \omega$) by Alas et al. [Topology Appl. 107 (2000) 259–273].

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1. Introduction

For an infinite cardinal κ and a family $\mathcal{A}$ of a set X, one says that $\mathcal{A}$ is κ-*independent* on X if for every finite $\mathcal{F} \subseteq \mathcal{A}$ and $f \in \{-1,+1\}^{\mathcal{F}}$ one has $|\bigcap_{A\in\mathcal{F}} A^{f(A)}| \geqslant \kappa$; here

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$A^{+1} = A$ and $A^{-1} = X \backslash A$. It is obvious from Zorn's Lemma that every κ-independent family on a set X expands to a maximal such family. This paper originated with a question posed by Eckertson [8]: Can there be, for $\kappa > \omega$, a family $\mathcal{A}$ on κ which is simultaneously maximal ω-independent and (maximal) κ-independent? We show in ZFC, contrary to our original intuitive belief, that such a family exists over every infinite κ (Theorem 3.6). Furthermore, according to Theorem 5.6, if $\kappa = \log(2^\kappa)$ then *every* suitably restricted maximal κ-independent family on a set of cardinality κ is maximal ω-independent.

In Section 4 we give a direct and elementary proof of this statement: For $\kappa \geqslant \omega$, the compact group $\{0,1\}^{2^\kappa}$ contains a dense subgroup D, every two points of which differ in 2^κ-many coordinates, such that each nonempty open $U \subseteq D$ satisfies $|U| = d(U) = |D| = d(D) = \kappa$. From this we are able to show (Corollary 5.4) that the same space $\{0,1\}^{2^\kappa}$ contains a dense subset E, every two points of which differ in 2^κ-many coordinates, such that (a) every nonempty open $U \subseteq E$ satisfies $|U| = d(U) = |E| = d(E) = \kappa$ and (b) no nonempty subset of E contains complementary (relatively) dense subsets. This result addresses another question posed by Eckertson [8], and is in consonance with related responses already obtained by Feng [10] and by Alas et al. [1].

2. Notation and topological preliminaries

All results here are in ZFC.

The symbols κ, λ and μ denote cardinal numbers, usually but not always infinite, and ω is the least infinite cardinal. For X a set we write $[X]^\kappa = \{A \subseteq X\colon |A| = \kappa\}$; the symbols $[X]^{\leqslant\kappa}$ and $[X]^{<\kappa}$ are defined analogously. For $\kappa \geqslant \omega$ we write $\log(\kappa) := \min\{\lambda\colon 2^\lambda \geqslant \kappa\}$. The symbols ξ and η denote ordinal numbers.

Of course spaces of the form $\{0,1\}^\kappa$ are understood to carry the usual product (compact, Hausdorff) topology, but in general, except where explicit notice is given to the contrary, we impose and assume no separation properties on the topological spaces we hypothesize.

The weight and density character of a space $X = (X, \mathcal{T})$ are denoted by the symbols $w(X)$ and $d(X)$, respectively. The *open density* $\mathrm{od}(X, \mathcal{T})$ and the *dispersion character* $\Delta(X, \mathcal{T})$ are respectively the cardinal numbers

$$\mathrm{od}(X, \mathcal{T}) = \min\bigl\{d(U)\colon \emptyset \neq U \in \mathcal{T}\bigr\} \quad \text{and}$$
$$\Delta(X, \mathcal{T}) = \min\bigl\{|U|\colon \emptyset \neq U \in \mathcal{T}\bigr\}.$$

Remark. Some authors (see, for example, [5,6]) have considered the "nowhere density number" $\mathrm{nwd}(X)$ of a space X defined as

$$\mathrm{nwd}(X) = \min\bigl\{|A|\colon A \subseteq X, \mathrm{int}_X \mathrm{cl}_X A \neq \emptyset\bigr\}.$$

It is clear that $\mathrm{od}(X) = \mathrm{nwd}(X)$ for all spaces X.

The family of clopen (= open-and-closed) subsets of a space X is denoted $\mathrm{clop}(X)$. The space X is said to be *clopen-separated* if for every $S \in [X]^2$ there is $C \in \mathrm{clop}(X)$ such that $|S \cap C| = 1$.

Lemma 2.1. *Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be topologies on a set X such that each nonempty $\mathcal{T}_1$-open set is $\mathcal{T}_0$-dense in X, and set $\mathcal{T} = \mathcal{T}_0 \vee \mathcal{T}_1$. Then* $\mathrm{od}(X, \mathcal{T}) \geqslant \mathrm{od}(X, \mathcal{T}_0)$.

Proof. It is enough to show that every $\mathcal{T}$-dense subset E of a set $U = U_0 \cap U_1 \in \mathcal{T}$ with $U_i \in \mathcal{T}_i$ is $\mathcal{T}_0$-dense in U_0, for then, choosing E and U so that $|E| = d(U, \mathcal{T}) = \mathrm{od}(X, \mathcal{T})$, we will have

$$\mathrm{od}(X, \mathcal{T}_0) \leqslant d(U_0, \mathcal{T}_0) \leqslant |E| = \mathrm{od}(X, \mathcal{T}).$$

Let $\emptyset \neq N \in \mathcal{T}_0$, say with $N \subseteq U_0$. Then $\emptyset \neq N \cap U_1 \in \mathcal{T}$ with $N \cap U_1 \subseteq U \in \mathcal{T}$, so since E is $\mathcal{T}$-dense in U we have $\emptyset \neq (N \cap U_1) \cap E \subseteq N \cap E$, as required. □

Remark 2.2. The hypotheses of Lemma 2.1 appear to be nonsymmetric in $\mathcal{T}_0$ and $\mathcal{T}_1$. This is illusory, however, since the condition that every nonempty $U_1 \in \mathcal{T}_1$ is $\mathcal{T}_0$-dense is equivalent to the condition that if $\emptyset \neq U_1 \in \mathcal{T}_1$ and $\emptyset \neq U_0 \in \mathcal{T}_0$ then $U_1 \cap U_0 \neq \emptyset$. Thus the conclusion of Lemma 2.1 may be strengthened to include also the inequality $\mathrm{od}(X, \mathcal{T}) \geqslant \mathrm{od}(X, \mathcal{T}_1)$.

Definition. A subspace F of a space X is *regular-closed* in X if F has the form $F = \overline{U}^X$ for some open subset U of X.

Definition. A subset D of a product space $\{0,1\}^\kappa$ is *maximally dispersed* in $\{0,1\}^\kappa$ if every two of its elements differ maximally—that is, if every two elements of D differ in κ-many coordinates.

For the reader's convenience we note that the condition that a subset D of the space $\{0,1\}^\kappa$ is maximally dispersed admits a group-theoretic formulation, as follows. Let

$$G = \Sigma_{<\kappa} = \bigl\{x \in \{0,1\}^\kappa \colon \bigl|\{\eta < \kappa \colon x_\eta \neq 0\}\bigr| < \kappa\bigr\}.$$

Then, a subset D of $\{0,1\}^\kappa$ is maximally dispersed if and only if D meets each coset of G at most once.

The following result is routine. We use it (only) in proving Corollary 5.4.

Lemma 2.3. *Let $K = \{0,1\}^\kappa$ with $\kappa \geqslant \omega$, let U be open in K, $U \neq \emptyset$, and let $F = \overline{U}^K$. Then*

(a) *there is a homeomorphism $\phi : F \twoheadrightarrow K$; and*
(b) *if $\kappa > \omega$ and D is dense and maximally dispersed in K, then $\phi : F \twoheadrightarrow K$ may be chosen so that $\phi[D \cap U]$ also is maximally dispersed in K.*

Proof. It is well known [9, 2.7.12] that F depends on countably many coordinates in the sense that there is $C \in [\kappa]^\omega$ such that $F = \pi_C[F] \times \{0,1\}^{\kappa \setminus C}$. Clearly $\pi_C[F]$ is itself regular-closed in the usual Cantor space $\{0,1\}^C$, so there is a homeomorphism $\psi : \pi_C[F] \twoheadrightarrow \{0,1\}^C$ (cf. [9, 6.2.A(c)], [17, 4.2.5], or [19, 30.4]). Then

$$\phi := \psi \times \mathrm{id} : \pi_C[F] \times \{0,1\}^{\kappa \setminus C} \twoheadrightarrow \{0,1\}^C \times \{0,1\}^{\kappa \setminus C} = \{0,1\}^\kappa$$

is a homeomorphism from F onto $\{0,1\}^\kappa$. It is clear that if $x, y \in D \cap U$ and $x_\eta \neq y_\eta$ for all $\eta \in J \in [\kappa]^\kappa$, then $(\phi(x))_\eta \neq (\phi(y))_\eta$ for all $\eta \in J \setminus C \in [\kappa]^\kappa$. □

Remark 2.4. The function ϕ just defined has no pleasing algebraic properties. If D, and perhaps U, are subgroups of K, we do not claim that ϕ may be chosen so that $\phi[D \cap U]$ is a subgroup of K. This remark will become relevant in Section 5, where of necessity the dense subgroups of groups of the form $\{0,1\}^\kappa$ given in Section 4 become replaced by dense subsets.

3. Independent families

For X a (fixed) set and $A \subseteq X$, we write $A^{+1} = A$ and $A^{-1} = X \backslash A$.

Definitions 3.1. Let κ be a not necessarily infinite cardinal, let X be a set and $\mathcal{A} \subseteq \mathcal{P}(X)$, and let $\mathcal{B}_{\mathcal{A}} = \{\bigcap_{A \in \mathcal{F}} A^{f(A)}\colon\ \mathcal{F} \in [\mathcal{A}]^{<\omega},\ f \in \{+1,-1\}^{\mathcal{F}}\}$. Then

(a) $\mathcal{A}$ is a *κ-independent* family on X if each $B \in \mathcal{B}_{\mathcal{A}}$ satisfies $|B| \geqslant \kappa$; and
(b) $\mathcal{A}$ is a *κ-separating* family on X if each $S \in [X]^2$ satisfies $|\{A \in \mathcal{A}\colon\ |A \cap S| = 1\}| \geqslant \kappa$;
(c) A *separating* family is a 1-separating family.

We note (i) every subfamily of a κ-independent family is a κ-independent family; (ii) a κ-independent family on X (with $\kappa \geqslant \omega$) is an ω-independent family; and (iii) for X and κ fixed, every κ-independent family on X extends (expands) by Zorn's Lemma to a maximal such family.

Two additional observations are in order.

Theorem 3.2. *Let $\mathcal{A}$ be an ω-independent family on a set X. Then*

(a) *$\emptyset \notin \mathcal{B}_{\mathcal{A}}$ and the family $\mathcal{B}_{\mathcal{A}} \cup \{\emptyset\}$ is closed under finite intersection*; *and*
(b) *if $\mathcal{A}_i \subseteq \mathcal{A}$ with $\mathcal{A}_0 \cap \mathcal{A}_1 = \emptyset$ and $B_i \in \mathcal{B}_{\mathcal{A}_i}$ for $i = 0, 1$, then $B_0 \cap B_1 \neq \emptyset$.*

Proof. (a) It is enough to note that if $B_i = \bigcap_{A \in \mathcal{F}_i} A^{f_i(A)} \in \mathcal{B}_{\mathcal{A}}$ for $i = 0, 1$ with $\mathcal{F}_i \in [\mathcal{A}]^{<\omega}$ and $f_i \in \{-1,+1\}^{\mathcal{F}_i}$, then $B_0 \cap B_1 = \emptyset$ if and only if there is $A \in \mathcal{F}_0 \cap \mathcal{F}_1$ such that $f_0(A) \neq f_1(A)$.

(b) Here $B_0 \cap B_1 \in \mathcal{B}_{\mathcal{A}}$, so in fact $|B_0 \cap B_1| \geqslant \omega$. $\square$

Remarks 3.3. Let $\kappa \geqslant \omega$ and let $\mathcal{A}$ be an ω-independent family on a set X. Then:

(a) It follows from Theorem 3.2(a) that the family $\mathcal{B}_{\mathcal{A}} \cup \{\emptyset\}$ is a basis for a topology $\mathcal{T}_{\mathcal{A}}$ on X; $\mathcal{T}_{\mathcal{A}}$ is a Hausdorff topology if and only if $\mathcal{A}$ is a separating family on X; and $\mathcal{A}$ is κ-independent on X if and only if $\Delta(X, \mathcal{T}_{\mathcal{A}}) \geqslant \kappa$.
(b) It is immediate from Theorem 3.2(b) that if $\mathcal{A}_i \subseteq \mathcal{A}$ with $\mathcal{A}_0 \cap \mathcal{A}_1 = \emptyset$, then every nonempty $B_1 \in \mathcal{T}_{\mathcal{A}_1}$ is $\mathcal{T}_{\mathcal{A}_0}$-dense in X.

Discussion 3.4. Eckertson [8, 2.7] asked whether for $|X| = \kappa > \omega$ there exists a family $\mathcal{A} \subseteq \mathcal{P}(X)$ which is both a maximal ω-independent family and a maximal κ-independent family on X? The authors' initial reaction was "No", for this reason: Given a κ-independent family $\mathcal{A}_0$ on X one may apply Zorn's Lemma to achieve a maximal κ-independent family $\mathcal{A}_\kappa \supseteq \mathcal{A}_0$ or, alternatively, a maximal ω-independent family $\mathcal{A}_\omega \supseteq \mathcal{A}_0$; but almost surely $\mathcal{A}_\kappa$ will not be maximal ω-independent, and $\mathcal{A}_\omega$ will no longer be κ-independent. Our first approach to Eckertson's problem has been to focus on the property $\mathrm{od}(X, \mathcal{T}_{\mathcal{A}_0}) \geqslant \kappa$, a property stronger than κ-independence (Remark 3.3(a)) which is enjoyed by some ω-independent families $\mathcal{A}_0$ (Theorem 3.6, Proof 1) and is preserved under passage to any maximal independent family $\mathcal{A} \supseteq \mathcal{A}_0$ (Theorem 4.5). An alternative approach, quite different in flavor, is also available (see Proof 2 of Theorem 3.6): Given a maximal κ-independent family $\mathcal{A}_0$ on κ with $|\mathcal{A}_0| = 2^\kappa$, one may replace the members of $\mathcal{A}_0$ by (slightly) smaller sets which constitute a new family of the desired sort; an analysis of the construction (see Discussion 3.7) shows that the revised family $\mathcal{A}$ also satisfies the condition $\mathrm{od}(X, \mathcal{T}_{\mathcal{A}}) \geqslant \kappa$.

Theorem 3.5. *Let $\kappa \geqslant \omega$, let X be a set and $\mathcal{A}_0$ an ω-independent family on X such that* $\mathrm{od}(X, \mathcal{T}_{\mathcal{A}_0}) \geqslant \kappa$*, and let $\mathcal{A}$ be an ω-independent family on X such that $\mathcal{A} \supseteq \mathcal{A}_0$. Then* $\mathrm{od}(X, \mathcal{T}_{\mathcal{A}}) \geqslant \kappa$.

Proof. Set $\mathcal{A}_1 = \mathcal{A} \backslash \mathcal{A}_0$. If $B_i \in \mathcal{B}_{\mathcal{A}_i}$ for $i = 0, 1$ then $B_0 \cap B_1 \neq \emptyset$ by Theorem 4.2(b), so Lemma 2.1 applies to give $\mathrm{od}(X, \mathcal{T}_{\mathcal{A}}) \geqslant \mathrm{od}(X, \mathcal{T}_{\mathcal{A}_0}) \geqslant \kappa$. □

Theorem 3.6. *Let $\kappa \geqslant \omega$. Then there is a family $\mathcal{A} \subseteq \mathcal{P}(\kappa)$ such that $|\mathcal{A}| = 2^\kappa$, $\mathcal{A}$ is 2^κ-separating, and $\mathcal{A}$ is both a maximal κ-independent family and a maximal ω-independent family on κ.*

Proof 1. Let $X = \mathrm{clop}(K)$ with $K := \{0, 1\}^\kappa$, for $t \in K$ let $X_t = \{C \in X\colon\ t \in C\}$, and set $\mathcal{A}_0 = \{X_t\colon\ t \in K\}$. Then $|X| = \kappa$ and the map $K \twoheadrightarrow \mathcal{A}_0 \subseteq \mathcal{P}(X)$ given by $t \to X_t$ is a bijection (so $|\mathcal{A}_0| = 2^\kappa$). For distinct $C(0), C(1) \in X$ each $t \in C(0) \Delta C(1) \in [K]^{2^\kappa}$ satisfies $X_t \cap \{C(0), C(1)\} = 1$, so $\mathcal{A}_0$ is 2^κ-separating on X. The family $\mathcal{A}_0$ is κ-independent on X, for if F_0, F_1 are disjoint finite subsets of K then there are κ-many $C \in X$ such that $F_1 \subseteq C$ and $F_0 \cap C = \emptyset$—that is, the set $B := (\bigcap_{t \in F_1} X_t) \backslash (\bigcup_{t \in F_0} X_t) \in \mathcal{B}_{\mathcal{A}_0}$ satisfies $|B| = \kappa$. It suffices to show $\mathrm{od}(X, \mathcal{T}_{\mathcal{A}_0}) = \kappa$ for then, according to Theorem 3.5, any (maximal) ω-independent family $\mathcal{A} \supseteq \mathcal{A}_0$ will satisfy $\Delta(X, \mathcal{T}_{\mathcal{A}}) \geqslant \mathrm{od}(X, \mathcal{T}_A) \geqslant \kappa$ (cf. Remark 3.3(a)); for this, since each $\mathcal{T}_{\mathcal{A}_0}$-basic set $B \in \mathcal{B}_{\mathcal{A}_0}$ is homeomorphic to X itself, it suffices to show that no $X' \in [X]^{<\kappa}$ is dense in X. Such X' is not a base for K (since $w(K) = \kappa$) so there are distinct $t, u \in K$ not separated by any element of X'; then with $C \in X$ chosen so that $t \in C$ and $u \notin C$ we have: $X_t \backslash X_u = X_t^{+1} \cap X_u^{-1}$ is a $\mathcal{T}_{\mathcal{A}_0}$-neighborhood of C which is disjoint from X'. □

Proof 2. Let $\mathcal{A}$ be a maximal κ-independent family on κ with $|\mathcal{A}| = 2^\kappa$, let $\phi : \mathcal{A} \twoheadrightarrow [\kappa]^{<\kappa}$ satisfy $|\phi^{-1}(S)| \geqslant \omega$ for all $S \in [\kappa]^{<\kappa}$, for $A \in \mathcal{A}$ let $A' = A \backslash \phi(A)$, and define $\mathcal{A}' := \{A'\colon\ A \in \mathcal{A}\}$. It is clear that $\mathcal{A}'$ is a maximal κ-independent family on κ. To see that $\mathcal{A}'$ is maximal ω-independent, we show for $Y \in \mathcal{P}(\kappa) \backslash \mathcal{A}'$ that $\mathcal{A}' \cup \{Y\}$ is not ω-independent. Since $Y \notin \mathcal{A}'$, there is $B' = \bigcap_{A \in \mathcal{F}} A'^{f(A')} \in \mathcal{B}_{\mathcal{A}'}$ (with $\mathcal{F} \in [\mathcal{A}]^{<\omega}$, $f \in \{-1, +1\}^{\mathcal{F}}$) such that $B' \cap Y \in [\kappa]^{<\kappa}$ or $B' \backslash Y \in [\kappa]^{<\kappa}$. In the former case, using the fact that $|\phi^{-1}(B' \cap Y)| \geqslant \omega$, there is $A \in \mathcal{A}$ such that $A \notin \mathcal{F}$ and $\phi(A) = B' \cap Y$, and then

$A' \cap B' \cap Y^{+1} = \emptyset$; similarly in the latter case there is $A \in \mathcal{A} \backslash \mathcal{F}$ such that $\phi(A) = B' \backslash Y$, and then $(A' \cap B') \cap Y^{-1} = \emptyset$. Thus in any case $\mathcal{A}' \cup \{Y\}$ is not ω-independent.

The map $A \to A'$ from $\mathcal{A}$ onto $\mathcal{A}'$ is bijective, so $|\mathcal{A}'| = |\mathcal{A}| = 2^\kappa$.

Using a trick of Eckertson [8, p. 2] one may inflect the elements of the family $\mathcal{A}'$ so that the new family $\mathcal{A}''$ is 2^κ-separated. Write $\mathcal{A}' = \{A'_\xi\colon \xi < 2^\kappa\}$ (faithfully indexed), write $[\kappa]^2 = \{x_\xi\colon \xi < 2^\kappa\}$ with each $x \in [\kappa]^2$ appearing 2^κ-many times, and set $A''_\xi := A'_\xi \cup \{\max x_\xi\} \backslash \{\min x_\xi\}$. Then $\mathcal{A}'' := \{A''_\xi\colon \xi < 2^\kappa\}$ is 2^κ-separated, and $\mathcal{A}''$ retains the desired cardinality and maximality properties already verified for $\mathcal{A}'$. □

Discussion 3.7. For applications below it is crucial that the argument of Proof 1 of Theorem 3.6 furnished on a set D with $|D| = \kappa$ a family $\mathcal{A} \subseteq \mathcal{P}(D)$ which is not only κ-independent but which satisfies the stronger property $\mathrm{od}(D, \mathcal{T}_{\mathcal{A}}) = \kappa$. It is interesting to note that the family $\mathcal{A}'$ (and hence $\mathcal{A}''$) of Proof 2 achieves this also. To see that no $S \in [\kappa]^{<\kappa}$ is dense in a $\mathcal{T}_{\mathcal{A}'}$-basic set $B' = \bigcap_{A \in \mathcal{F}} A'^{f(A')} \in \mathcal{B}_{\mathcal{A}'}$, choose $A \in \mathcal{A} \backslash \mathcal{F}$ such that $\phi(A) = S$; then $\emptyset \neq B' \cap A' \in \mathcal{B}_{\mathcal{A}'} \subseteq \mathcal{T}_{\mathcal{A}'}$, and $B' \cap A' \cap S = \emptyset$.

Remark 3.8. Families $\mathcal{A}$ as in Theorem 3.6 are of course maximal μ-independent for every $\mu \in [\omega, \kappa]$.

4. Dense subgroups of $\{0, 1\}^\kappa$

Remark 4.1. The group $D := \{0, 1\}^\kappa$ is compact, so each of its subgroups is *totally bounded* (in the terminology of some authors: *precompact*) in the sense that for each nonempty open $U \subseteq D$ there is $F \in [D]^{<\omega}$ such that $D = F + U$. From this it follows easily for such U that $|D| = |U|$ (hence $|D| = \Delta(D)$), and also that $d(D) = d(U)$ (hence $d(D) = \mathrm{od}(D)$).

Next, in parallel with Proof 1 of Theorem 3.6, we give a variant of a familiar argument of Fichtenholz and Kantorovitch [11] as extended by Hausdorff [13]. See also [15, 24.8], [12, 9.1], [7, §3] and [9, 2.3.15, 3.6.F] for related treatments.

Theorem 4.2. *Let κ and λ be infinite cardinals for which there is a clopen-separated space X such that $|X| = \Delta(X) = \kappa$ and*

$$\lambda = \min\bigl\{|\mathcal{C}|\colon \mathcal{C} \subseteq \mathrm{clop}(X) \text{ and } \mathcal{C} \text{ separates points of } X\bigr\}.$$

Then there is a dense subgroup D of $\{0, 1\}^\kappa$ such that D is maximally dispersed and $|D| = \mathrm{od}(D) = \lambda$.

Proof. Let X and $\mathcal{C} \in [\mathrm{clop}(X)]^\lambda$ be chosen as indicated. We assume without loss of generality, augmenting $\mathcal{C}$ if necessary, that $\mathcal{C}$ is closed under complementation and finite unions (and hence under finite intersections); this can be achieved without increasing the cardinal number $|\mathcal{C}|$. For $S \subseteq X$ let χ_S denote the characteristic function of S, and set $D := \{\chi_C\colon C \in \mathcal{C}\}$. It is routine to verify that the map $\mathcal{C} \twoheadrightarrow D$ given by $C \to \chi_C$ is an isomorphism of the group $(\mathcal{C}, \Delta)$ onto the subgroup D of $\{0,1\}^X$; in particular we have $|D| = |\mathcal{C}| = \lambda$.

If F_0 and F_1 are disjoint finite subsets of X, then since $\mathcal{C}$ separates points of X and is closed under the usual Boolean operations, there is $C \in \mathcal{C}$ such that $F_1 \subseteq C$ and $F_0 \cap C = \emptyset$. Then $\chi_C \equiv 1$ on F_1, and $\chi_C \equiv 0$ on F_0. This shows that D is dense in $\{0,1\}^\kappa$.

To complete the proof it now suffices, according to Remark 4.1, to show that $d(D) = \lambda$. Suppose there is $\mathcal{C}' \in [\mathcal{C}]^{<\lambda}$ such that $E := \{\chi_C\colon\ C \in \mathcal{C}'\}$ is dense in D. We again suppose without loss of generality, augmenting $\mathcal{C}'$ if necessary while keeping its cardinality unchanged, that $\mathcal{C}'$ is closed under finite intersection and complementation. Since $\mathcal{C}'$ does not separate points of X, there are distinct $t, u \in X$ such that $\chi_C(t) = \chi_C(u)$ for each $\chi_C \in E$. Choosing $B \in \mathcal{C}$ so that $t \in B$ and $u \notin B$ we have $\chi_B \in D$ with $\chi_B(t) = 1 \neq 0 = \chi_B(u)$. Then clearly $\chi_B \notin \overline{E}^D$. □

Corollary 4.3. *Let $\kappa \geqslant \omega$. Then there is a maximally dispersed, dense subgroup D of $\{0,1\}^{2^\kappa}$ such that $|D| = d(D) = \mathrm{od}(D) = \kappa$.*

Proof. The space $X := \{0,1\}^\kappa$ is clopen-separated with $|X| = \Delta(X) = 2^\kappa$, and every point-separating $\mathcal{C} \subseteq \mathrm{clop}(X)$ satisfies $|\mathcal{C}| = |\mathrm{clop}(X)| = \kappa$. Thus Theorem 4.2 applies (with (κ, λ) replaced by $(2^\kappa, \kappa)$). □

Discussion 4.4. The authors did not attempt to characterize those pairs (λ, κ) of infinite cardinals which satisfy the hypotheses of Theorem 4.2. (We do note in passing that with X and $\mathcal{C}$ as in Theorem 4.2 the map $X \to \mathcal{P}(\mathcal{C})$ given by $x \to \{C \in \mathcal{C}\colon\ x \in C\}$ is injective, so that $2^\lambda = |\mathcal{P}(\mathcal{C})| \geqslant |X| = \kappa$—that is, $\lambda \geqslant \log(\kappa)$. Clearly since X is clopen-separated some $\mathcal{D} \subseteq \mathrm{clop}(X)$ with $|\mathcal{D}| = |[X]^2| = |X| = \kappa$ separates points of X, so also $\lambda \leqslant \kappa$.)

Remark 4.5. If the infinite cardinals κ and λ satisfy $\log(2^\kappa) \leqslant \lambda \leqslant \kappa$, then $2^\lambda = 2^\kappa$ (since $2^\kappa \leqslant 2^{\log(2^\kappa)} \leqslant 2^\lambda \leqslant 2^\kappa$). Thus Corollary 4.3 has the following consequence.

Corollary 4.6. *Let $\kappa \geqslant \omega$. For every $\lambda \in [\log(2^\kappa), \kappa]$ there is a maximally dispersed, dense subgroup D of $\{0,1\}^{2^\kappa}$ such that $|D| = d(D) = \mathrm{od}(D) = \lambda$.*

5. Consequences concerning resolvability

Following Hewitt [14], we say that a space $(X, \mathcal{T})$ is *resolvable* if X admits a pair of complementary dense subsets; a space with no nonempty resolvable subspace is said to be *hereditarily irresolvable*. To catch the flavor of several of the many results in the literature relating to resolvability and its generalizations, the interested reader might consult [14,2,5, 6,8,10,3], and some of the papers cited therein.

Discussion 5.1. In what follows we begin with a topological group $(D, \mathcal{T}) = (D, \mathcal{T}_{\mathcal{A}_0})$ as in Theorem 4.2, with $\mathcal{A}_0$ defined as in Proof 1 of Theorem 3.6: $|D| = \kappa$, $\mathcal{A}_0 \subseteq \mathcal{P}(D)$, $|\mathcal{A}_0| = 2^\kappa$, $\mathcal{A}_0$ is 2^κ-separating and κ-independent, and $\mathrm{od}(D) = \mathrm{od}(D, \mathcal{T}_{\mathcal{A}_0}) = \kappa$. We choose $\mathcal{A}$ so that $\mathcal{A}_0 \subseteq \mathcal{A} \subseteq \mathcal{P}(D)$ and $\mathcal{A}$ is both maximal ω-independent and (necessarily maximal) κ-independent. Let K be the space $\{0,1\}^{2^\kappa}$ with its usual product topology, and define $e: D \to \{0,1\}^{\mathcal{A}} = K$ by the rule

$$(ex)_A = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A \end{cases}$$

for $x \in D$, $A \in \mathcal{A}$.

Lemma 5.2. *The function e is a homeomorphism from $(D, \mathcal{T}_{\mathcal{A}})$ into K.*

Proof. Since $\mathcal{A}$ is separating, e is an injection into K.

Let $\mathcal{B}$ be the trace on $e[D]$ of the canonical basis for K. To see that e exhibits a bijection from $\mathcal{B}_{\mathcal{A}}$ onto $\mathcal{B}$, note first that if $B = \bigcap_{A \in \mathcal{F}} A^{f(A)} \in \mathcal{B}_{\mathcal{A}}$ with $\mathcal{F} \in [\mathcal{A}]^{<\omega}$ and $f \in \{-1, +1\}^{\mathcal{F}}$, then

$$\begin{aligned} e[B] = &\bigcap \bigl\{\pi_A^{-1}\bigl(\{1\}\bigr):\ A \in f^{-1}\bigl(\{1\}\bigr)\bigr\} \\ &\cap \bigcap \bigl\{\pi_A^{-1}\bigl(\{0\}\bigr):\ A \in f^{-1}\bigl(\{-1\}\bigr)\bigr\} \cap D \in \mathcal{B}; \end{aligned}$$

and conversely each set $U = \{0\}_{F_0} \times \{1\}_{F_1} \times \{0,1\}^{\mathcal{A} \setminus (F_0 \cup F_1)} \in \mathcal{B}$ (with $F_i \in [\mathcal{A}]^{<\omega}$) has the form $U = e[B]$ for

$$B = \Bigl(\bigcap_{A \in \mathcal{F}_0} A^{-1}\Bigr) \cap \Bigl(\bigcap_{A \in \mathcal{F}_1} A^{+1}\Bigr) \in \mathcal{B}_{\mathcal{A}}. \qquad \square$$

Theorem 5.3. *Let $\kappa \geqslant \omega$ and let $K = \{0,1\}^{2^\kappa}$. Then there is a maximally dispersed, dense, irresolvable subset D of K such that $|D| = d(D) = \mathrm{od}(D) = \kappa$.*

Proof. Again let D and $\mathcal{A}_0 \subseteq \mathcal{P}(D)$ be as in 5.1 and define $e: D \to \{0,1\}^{\mathcal{A}} = \{0,1\}^{2^\kappa} = K$ with $\mathcal{A}_0 \subseteq \mathcal{A} \subseteq \mathcal{P}(D)$. Using Lemma 5.2, we suppress mention of e and we write $D = (D, \mathcal{T}_{\mathcal{A}}) \subseteq K$.

D is dense in K. Every basic set $U \subseteq K$ satisfies $U \cap D \in \mathcal{B}_{\mathcal{A}}$, so $|U \cap D| = \kappa$ since $\mathcal{A}$ is κ-independent.

$|D| = \mathrm{od}(D) = \kappa$. We note for clarity that from $\mathcal{T}_{\mathcal{A}_0} \subseteq \mathcal{T}_{\mathcal{A}}$ follows

$$\kappa = |D| \geqslant d(D) \geqslant d(D, \mathcal{T}_{\mathcal{A}_0}) = \kappa,$$

so $d(D) = \kappa$. This remark is not strictly necessary, however: it is enough to show directly that $\mathrm{od}(D) = \kappa$. For this, define $\mathcal{A}_1 := \mathcal{A} \setminus \mathcal{A}_0$. It then follows from Remark 3.3(b) that every nonempty $B_1 \in \mathcal{T}_{\mathcal{A}_1}$ is $\mathcal{T}_{\mathcal{A}_0}$-dense in D. Then

$$\kappa = |D| \geqslant \mathrm{od}(D, \mathcal{T}_{\mathcal{A}}) \geqslant \mathrm{od}(D, \mathcal{T}_{\mathcal{A}_0}) = \kappa$$

by Lemma 2.1.

D is maximally dispersed in $K = \{0,1\}^{2^\kappa} = \{0,1\}^{\mathcal{A}}$. The family $\mathcal{A}_0$ is 2^κ-separating on D, and $\mathcal{A} \supseteq \mathcal{A}_0$.

D is irresolvable. Suppose there is $E \subseteq D$ such that E and $D\backslash E$ are both dense in D. Then $|E \cap C| = |(D\backslash E) \cap C| = \kappa$ for each $C \in \mathcal{B}_{\mathcal{A}}$ since $\mathrm{od}(D) = \kappa$, so $\mathcal{A} \cup \{E\}$ (and similarly $\mathcal{A} \cup \{D\backslash E\}$) is κ-independent. Thus $E \in \mathcal{A}$ and $D\backslash E \in \mathcal{A}$ by maximality of $\mathcal{A}$, and we have the contradiction $|E \cap (D\backslash E)| = \kappa$. □

We note in passing that the topology $\mathcal{T}_{\mathcal{A}_1}$ introduced in the proof above may fail to be a Hausdorff topology. Of course Lemma 2.1 applies as indicated (cf. the introductory sentences to Section 2). We note also that the space $(D, \mathcal{T}_{\mathcal{A}_0})$ of 4.2 and 5.1 is a topological group, but the topology $\mathcal{T}_{\mathcal{A}}$ on D is no longer the topology inherited from $\{0,1\}^{\mathcal{A}_0}$. Indeed the space $(D, \mathcal{T}_{\mathcal{A}})$ is irresolvable, but it is known [16] (see also [6] for additional details) that every totally bounded topological group is resolvable.

With Theorem 5.3 now available, its consequence 5.4 is achieved through standard arguments. In the special case $\kappa = \omega$ and without the word "hereditarily", most of this is given (by a quite different argument) by Alas et al. [1, Theorem 2.3].

Corollary 5.4. *Let $\kappa \geqslant \omega$. There is a maximally dispersed, dense, hereditarily irresolvable subset E of $K := \{0,1\}^{2^\kappa}$ such that $|E| = d(E) = \mathrm{od}(E) = \kappa$.*

Proof. By Theorem 5.3 there is a maximally dispersed, dense, irresolvable subset D of K such that $|D| = d(D) = \mathrm{od}(D) = \kappa$. Using an idea of Hewitt [14, Theorem 28], let Y be the union of all resolvable subsets of D and set $E := D\backslash \overline{Y}^D$. The space Y is resolvable [4], so $\overline{Y}^D$ is resolvable. Clearly no (nonempty) subspace of E is resolvable, and since D is irresolvable we have $E \neq \emptyset$. Now choose U open in K such that $U \cap D = E$, and set $F := \overline{E}^K = \overline{U}^K$. The required homeomorphism $\phi : F \twoheadrightarrow K$ is given by Lemma 2.3, with 2^κ replacing κ. □

Remarks 5.5.

(a) Corollary 5.4, answering another question of Eckertson [8], coincides in its essentials with the special case $n = 1$ of the answer of Feng [10] to the same question. Feng's approach to this question of Eckertson is informative, and it differs significantly from ours: He uses the theory of the Bohr compactification of a (discrete) abelian group.
(b) The elementary argument given in Remark 4.6 allows a statement (superficially) stronger than Corollary 5.4, as follows.

Let $\kappa \geqslant \omega$. For every $\lambda \in [\log(2^\kappa), \kappa]$ there is a maximally dispersed, dense, hereditarily irresolvable subset E of $\{0,1\}^{2^\kappa}$ such that $|E| = d(E) = \mathrm{od}(E) = \lambda$.

It is not difficult to show (for $\kappa > \omega$) that some (maximal) ω-independent family on κ is not κ-independent. The question whether every maximal κ-independent family of size 2^κ on κ is a maximal ω-independent family on κ is suitably settled by our next result.

Theorem 5.6. *Let $\kappa \geqslant \omega$. Then the following statements are equivalent.*

(a) $\log(2^\kappa) = \kappa$;
(b) *no dense subgroup* D *of the space* $\{0,1\}^{2^\kappa}$ *satisfies* $\mathrm{od}(D) < \kappa$; *and*
(c) *every* 2^κ*-separating, maximal* κ*-independent family on* κ *is a maximal* ω*-independent family.*

Proof. (a) $\Rightarrow$ (b). Together with the above-cited Hewitt–Marczewski–Pondiczery theorem, a counterexample D to (b) would yield the relations

$$\log\bigl(2^\kappa\bigr) = d\bigl(\{0,1\}^{2^\kappa}\bigr) \leqslant d(D) = \mathrm{od}(D) < \kappa.$$

(b) $\Rightarrow$ (a). We have noted in Corollary 4.6 and Remark 4.5 that if $\log(2^\kappa) \leqslant \lambda \leqslant \kappa$ then $2^\lambda = 2^\kappa$ and there is a dense subgroup D of the space $\{0,1\}^{2^\kappa} = \{0,1\}^{2^\lambda}$ such that $|D| = \mathrm{od}(D) = \lambda$. Thus (b) fails if (a) fails.

(b) $\Rightarrow$ (c). Let $\mathcal{A} \subseteq \mathcal{P}(D)$ with $|D| = \kappa$ be a family as hypothesized in (c), and using the homeomorphism $e : (D, \mathcal{T}_\mathcal{A}) \to K = \{0,1\}^\mathcal{A} = \{0,1\}^{2^\kappa}$ write $(D, \mathcal{T}_\mathcal{A}) \subseteq K$ with D dense in K and with $\mathcal{T}_\mathcal{A}$ the topology inherited from K. Each dense subset E of an open subset U of D has $\langle E\rangle$ dense in $\langle U\rangle$, so from (b) follows

$$\kappa = \mathrm{od}\bigl(\langle D\rangle\bigr) \leqslant \mathrm{od}(D) \leqslant |D| = \kappa.$$

It suffices to prove that if $A \subseteq D$ and $\mathcal{A} \cup \{A\}$ is ω-independent then $A \in \mathcal{A}$. For such A we have $A \cap B \neq \emptyset$ for each $B \in \mathcal{B}_\mathcal{A}$, so A is dense in $\mathcal{T}_\mathcal{A}$ and hence $|A \cap B| \geqslant \mathrm{od}(D) = \kappa$. Similarly $|(D\backslash A) \cap B| = \kappa$ for each $B \in \mathcal{B}_\mathcal{A}$. Thus $\mathcal{A} \cup \{A\}$ is κ-independent so indeed $A \in \mathcal{A}$.

(c) $\Rightarrow$ (a). Suppose that (a) fails, write $\lambda := \log(2^\kappa) < \kappa$, and let X_λ and X_κ be disjoint sets of cardinality λ and κ, respectively. Then by Theorem 3.6, for $\mu \in \{\lambda, \kappa\}$ there is a family $\mathcal{A}_\mu \subseteq \mathcal{P}(X_\mu)$ such that $|\mathcal{A}_\mu| = 2^\mu = 2^\kappa$, $\mathcal{A}_\mu$ is 2^μ-separating, and $\mathcal{A}_\mu$ is both a maximal μ-independent family on X_μ and a maximal ω-independent family on X_μ. Fix $A \in \mathcal{A}_\lambda$, write $\mathcal{A}_\lambda\backslash\{A\} = \{A^\lambda_\eta\colon \eta < 2^\kappa\}$ and $\mathcal{A}_\kappa = \{A^\kappa_\eta\colon \eta < 2^\kappa\}$, and define $\mathcal{A} := \{A_\eta\colon \eta < 2^\kappa\} \subseteq \mathcal{P}(X)$ with $X := X_\lambda \cup X_\kappa$ and $A_\eta := A^\lambda_\eta \cup A^\kappa_\eta$. Then $|\mathcal{A}| = 2^\kappa$, and $\mathcal{A}$ is κ-independent on X since $\mathcal{A}_\kappa$ is κ-independent on X_κ. It is clear that $A \notin \mathcal{A}$ and $\mathcal{A} \cup \{A\}$ is λ-independent on X, so $\mathcal{A}$ is not a maximal λ-independent family on X (hence, not a maximal ω-independent family on X). We claim that $\mathcal{A}$ is a maximal κ-independent family on X. Suppose that $H \subseteq X$ and $\mathcal{A} \cup \{H\}$ is κ-independent on X. Then $\mathcal{A} \cup \{H \cap X_\kappa\}$ is κ-independent on X; thus $\mathcal{A}_\kappa \cup \{H \cap X_\kappa\}$ is κ-independent on X_κ so $H \cap X_\kappa \in \mathcal{A}_\kappa$ and there is $\eta < 2^\kappa$ such that $H \cap X_\kappa = A^\kappa_\eta$. It then follows that $H = A_\eta = A^\kappa_\eta \cup A^\lambda_\eta \in \mathcal{A}$, for if $H = A^\kappa_\eta \cup B$ with $B \subseteq X_\lambda$ and $B \neq A^\lambda_\eta$ then we have simultaneously $|H \cap (X\backslash A_\eta)| \leqslant \lambda < \kappa$ (since $H \cap (X\backslash A_\eta) \subseteq X_\lambda$) and $|H \cap (X\backslash A_\eta)| \geqslant \kappa$ (since $A_\eta \in \mathcal{A}$ and $\mathcal{A} \cup \{H\}$ is κ-independent). □

It is immediate from work of Shelah [18] that it is consistent with the axioms of ZFC that some maximal ω-independent family $\mathcal{A}$ on ω satisfies $|\mathcal{A}| < 2^\omega$. We have not successfully approached the analogous question for cardinals $\kappa > \omega$. We pose two questions.

Question 5.7. Given $|X| = \kappa > \omega$, is it consistent with the axioms of ZFC that some maximal κ-independent family $\mathcal{A}$ on X satisfies $|\mathcal{A}| < 2^\kappa$?

Question 5.8. Evidently Lusin's Hypothesis $2^{\omega} = 2^{\omega_1}$ implies that there is on ω_1 a maximal ω_1-independent family $\mathcal{A}$ such that $|A| = 2^{\omega}$. Does the converse implication hold?

Discussion 5.9. Let us say, in parallel with earlier definitions, that a family $\mathcal{A} \subseteq \mathcal{P}(X)$ is

(i) *maximally independent* if each $B \in \mathcal{B}_{\mathcal{A}}$ satisfies $|B| = |X|$; and
(ii) *maximally separating* if each $S \in [X]^2$ satisfies $|\{A \in \mathcal{A}\colon |A \cap S| = 1\}| = 2^{|X|}$.

As indicated above, the results of this paper have been in ZFC (with no additional axioms). Our final result, Theorem 5.10, as well as Question 5.12, are of slightly different flavor.

Theorem 5.10. *If GCH holds, then the following* (*equivalent*) *conditions are satisfied.*

(a) *every* $\kappa \geqslant \omega$ *satisfies* $\log(2^{\kappa}) = \kappa$; *and*
(b) *every maximally separating, maximally independent family on an infinite set is a maximal* ω*-independent family.*

Proof. It is obvious that GCH $\Rightarrow$ (a), while the equivalence (a) $\Leftrightarrow$ (b) is immediate from Theorem 5.6. □

Discussion 5.11. Given $\kappa \geqslant \omega$, let $\mathcal{H}(\kappa)$ denote the set of cardinals λ such that there is a dense, hereditarily irresolvable subset E of $\{0,1\}^{2^{\kappa}}$ such that $|E| = \mathrm{od}(E) = \lambda$. We noted in Corollary 5.4 that every λ such that $2^{\lambda} = 2^{\kappa}$ satisfies $\lambda \in \mathcal{H}(\kappa)$; in particular, $[\log(2^{\kappa}), \kappa] \subseteq \mathcal{H}(\kappa)$. It is a theorem of Hewitt [14, Theorem 42] (cf. also Ceder [2]) that every space X such that $\Delta(X) \geqslant w(X) \geqslant \omega$ is resolvable; thus in particular $2^{\kappa} \notin \mathcal{H}(\kappa)$. These observations leave open the following question.

Question 5.12. Are there models of ZFC with cardinal pairs $\{\lambda, \kappa\}$ such that $2^{\lambda} > 2^{\kappa}$ and $\lambda \in \mathcal{H}(\kappa)$?

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