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Cross sections and homeomorphism classes of Abelian groups equipped with the Bohr topology

W.W. Comfort [a], Salvador Hernández [b,1], F. Javier Trigos-Arrieta [c,*]

[a] *Department of Mathematics, Wesleyan University, Middletown, CT 06459, USA*
[b] *Departamento de Matemáticas, Universitat Jaume I, 12071 Castellón, Spain*
[c] *Department of Mathematics, California State University Bakersfield, Bakersfield, CA 93311, USA*

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Abstract

A closed subgroup H of a topological group G is a *ccs-subgroup* if there is a continuous cross section from G/H to G—that is, a continuous function Γ such that $\pi \circ \Gamma = \mathrm{id}|_{G/H}$ (with $\pi : G \to G/H$ the natural homomorphism).

The symbol $G^{\#}$ denotes an Abelian group G with its Bohr topology, i.e., the topology induced by $\mathrm{Hom}(G, \mathbb{T})$.

A topological group H is *an absolute ccs-group*(#) (respectively, *an absolute retract*(#)) if H is a ccs-subgroup (respectively, is a retract) in every group of the form $G^{\#}$ containing H as a (necessarily closed) subgroup. One then writes $H \in \mathrm{ACCS}(\#)$ (respectively, $H \in \mathrm{AR}(\#)$).

Theorem 1. *Every ccs-subgroup H of a group of the form $G^{\#}$ is a retract of $G^{\#}$* (*and $G^{\#}$ is homeomorphic to $(G/H)^{\#} \times H^{\#}$*)*; hence* $\mathrm{ACCS}(\#) \subseteq \mathrm{AR}(\#)$.

Theorem 2. $H^{\#} \in \mathrm{ACCS}(\#)$ (*respectively, $H^{\#} \in \mathrm{AR}(\#)$*) *if $H^{\#}$ is a ccs-subgroup of its divisible hull* $(\mathrm{div}(H))^{\#}$ (*respectively, $H^{\#}$ is a retract of* $(\mathrm{div}(H))^{\#}$).

Theorem 3.

(a) *Every cyclic group is in* $\mathrm{ACCS}(\#)$.

(b) *The classes* $\mathrm{ACCS}(\#)$ *and* $\mathrm{AR}(\#)$ *are closed under finite products.*

(c) *Not every Abelian group is in* $\mathrm{ACCS}(\#)$.

The following corollary to Theorem 3 answers a question of Kunen:

Corollary. *The spaces* $(\mathrm{div}(\mathbb{Z}))^{\#}$ *and* $((\mathrm{div}(\mathbb{Z})/\mathbb{Z}) \times \mathbb{Z})^{\#}$ *are homeomorphic.*

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* Corresponding author.

E-mail addresses: wcomfort@wesleyan.edu (W.W. Comfort), hernande@mat.uji.es (S. Hernández), jtrigos@cs.csubak.edu (F. Javier Trigos-Arrieta).

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1. Introduction

For simplicity, and because our methods are applicable for the most part only to Abelian groups, all groups considered in this paper are Abelian. We note explicitly however, that all statements in Definition 4, Remark 5 and Section 2 make sense or are true without this restriction. Of course when G is not Abelian and its closed subgroup H is perhaps not normal, the quotient space G/H must be interpreted as (say) the space of left cosets topologized as usual.

Every group G can be equipped with its largest totally bounded group topology. Also known as the *Bohr topology* of (discrete) G, this can be realized as the topology induced on G by the group $\mathrm{Hom}(G,\mathbb{T})$ of homomorphisms from G into the usual circle group $\mathbb{T}$. Since $\mathrm{Hom}(G,\mathbb{T})$ separates points of G, the Bohr topology is a Hausdorff group topology and hence a Tychonoff topology.

Following van Douwen [14], who initiated the first formal and extensive study of these groups *qua* topological spaces, we denote the group G in its Bohr topology by $G^\#$. For simplicity we call a topological group of the form $G^\#$ a *#-group*, and we denote the class of all #-groups by the symbol #. Contributions to the theory of #-groups have been given by some of the present authors [3,4,9,27,28] with co-authors [8], by Hart and van Mill [19], and by Protasov [24]. It stretches the truth only slightly to assert that, except for the standing Abelian hypothesis, proofs in these articles (including [14]) of results concerning #-groups make no use whatever of specific algebraic properties; the arguments turn strictly on combinatorics related to the cardinality of the groups in question. Indeed, van Douwen [13] posed the following bold and beautiful question:

Question 1. If the groups G_1 and G_2 have the same cardinality, must the spaces $G_1^\#$ and $G_2^\#$ be homeomorphic?

In a preliminary response in the positive direction, Trigos-Arrieta [26, 6.33 and 6.36] noticed that if G contains a subgroup H of index $n \in \mathbb{N}$ such that $G^\#$ and $H^\#$ are homeomorphic, then $G^\#$ is homeomorphic to $G^\# \times \mathbb{Z}_n$, where $\mathbb{Z}_n$ stands for the discrete cyclic group of order n, thus showing that non-isomorphic groups of arbitrary infinite cardinal do exist with homeomorphic #-spaces. Using the fact that every nondiscrete countable infinite homogeneous space is homeomorphic to each of its nonempty clopen subsets (cf. van Douwen [12, 1.4], or Comfort and van Mill [5, 1.1]), Hart and Kunen [18], have generalized Trigos-Arrieta's result, showing for every infinite group G that $G^\#$ and $G^\# \times D$ are homeomorphic for every finite or countably infinite discrete space D. Nevertheless Kunen [23], and independently Dikranjan and Watson [11], have given examples of *torsion* groups with the same cardinality yielding nonhomeomorphic #-spaces. Thus Question 1 is solved in the negative. Much remains unknown, however, even among groups of countable cardinality. The arguments of [23,11] leave, for example, the following questions untouched: Which if any of the spaces $\mathbb{Z}^\#$, $(\mathbb{Z}\times\mathbb{Z})^\#$, $(\mathrm{div}(\mathbb{Z}))^\#$, $(\mathrm{div}(\mathbb{Z})/\mathbb{Z})^\#$, $((\mathrm{div}(\mathbb{Z})/\mathbb{Z})\times\mathbb{Z})^\#$, $(\bigoplus_\omega \mathbb{Z})^\#$, $(\bigoplus_\omega \{0,1\})^\#$ are homeomorphic? It seems appropriate, accordingly, to replace Question 1 with the following (imperfectly posed) open-ended question.

Question 2. How do algebraic properties of the groups G affect the homeomorphism classes of the spaces $G^\#$?

Given groups G and H, Kunen [23] writes $G\sim H$ in case there are subgroups G_1 of G, and H_1 of H, each of finite index, such that G_1 and H_1 are isomorphic. He remarked that if $G\sim H$ then $G^\#$ and $H^\#$ are homeomorphic, and he asked if the converse holds. Corollary 26(a) *infra* showing that the spaces $(\mathrm{div}(\mathbb{Z}))^\#$ and $((\mathrm{div}(\mathbb{Z})/\mathbb{Z})\times\mathbb{Z})^\#$ are homeomorphic, simultaneously contributes to Question 2 above and shows that the answer to Kunen's question is “not always”.

If H has finite index in G, then $H^\#$ is clopen in $G^\#$, hence is a retract of $G^\#$. This phenomenon led van Douwen [14] to pose also the following question.

Question 3. Is every subgroup H of a group $G^\#$ a retract of $G^\#$?

In the present paper we also contribute toward Question 3 by identifying large classes of groups which are retracts wherever they are embedded. Our approach is *via* the following definition.

Definition 4. Let H be a closed subgroup of a (not necessarily Abelian) topological group G, and let $\pi: G\to G/H$ be the natural map.

(a) A *continuous cross section for G/H* is a continuous map $\Gamma: G/H\to G$ such that $\pi\circ\Gamma = \mathrm{id}|_{G/H}$;

(b) If G/H has a continuous cross section, then H is a *ccs-subgroup* of G.

Remark 5. According to alternative common terminology, when Γ is a continuous cross section for G/H the function $\Gamma\circ\pi: G\to G$ is called a *continuous selection*; and the set $\Gamma[G/H]$ is a *transversal set*.

We prove (Theorem 8) that if H is a ccs-subgroup of a topological group G, then G is homeomorphic to $(G/H)\times H$ and H is a retract of G. We show also that $H^\#$ is a ccs-subgroup of every enveloping #-group if and only if $H^\#$ is a ccs-subgroup of its divisible hull $(\mathrm{div}(H))^\#$ (Theorem 19). Since every finitely generated group is a ccs-subgroup of every enveloping #-group (Corollary 27), we obtain the results just cited, namely that $(\mathrm{div}(\mathbb{Z})/\mathbb{Z})^\#\times\mathbb{Z}^\#$ is homeomorphic to $(\mathrm{div}(\mathbb{Z}))^\#$ and $\mathbb{Z}^\#$ is a retract of $(\mathrm{div}(\mathbb{Z}))^\#$. Finally in Section 5, using techniques and arguments drawn from Kunen [23], we give an example of a group which is not an absolute ccs-group(#) (as defined in the abstract).

A question from [14] closely related to Question 3, whether every closed subset of a countable group of the form $G^\#$ is a retract, was solved in the negative by Gladdines

[17] (taking for G the countable Boolean group $\bigoplus_\omega\{0,1\}$). Already van Douwen [14] had shown that every uncountable G contains an uncountable, closed, nonretract, but apparently it remains unknown whether every group, even if assumed countable, contains a closed and countable nonretract.

Again for emphasis: Our groups are Abelian and our spaces are completely regular and Hausdorff, unless it is explicitly stated otherwise. $\mathbb{Q}$ is defined as the divisible hull of $\mathbb{Z}$. Undefined topological terms can be found in the monograph Engelking [15].

Remarks 6. Several of our results were presented at the *13th Summer Conference on General Topology and Applications* in Mexico City, by the third-listed author, and at the *Canadian Mathematical Society Winter Meeting* (*Special Session on Set-theoretic Topology*) (http://at.yorku.ca/cgi-bin/amca/cabr-15), by the first-listed author. See also our preprint in the *Topology Atlas* (http://at.yorku.ca/v/a/a/a/34.htm).

2. Arbitrary ccs-subgroups

Remark 7. Assume that G is a (not necessarily Abelian) topological group with a closed (not necessarily normal) subgroup H such that G/H admits a continuous cross section Γ. Then it may be assumed that $\Gamma(H) = 1_G$; indeed, to achieve this it is enough to replace Γ by its translation $xH \mapsto \Gamma(xH)\cdot(\Gamma(H))^{-1}$.

Theorem 8. *Let H be a* (*not necessarily normal*) *ccs-subgroup of a* (*not necessarily Abelian*) *topological group G. Then*

(a) *there is a homeomorphism Φ of G onto $(G/H)\times H$ which restricted to H takes $h\mapsto(H,h)$; and*

(b) *H is a retract of G.*

Proof. Let $\Gamma: G/H \to G$ be a continuous cross section taking H to 1_G, and define $\Theta:(G/H)\times H \to G$ by the rule $\Theta(\pi(x),h) = \Gamma(\pi(x))\cdot h$. Then Θ is continuous since it is the product of two continuous functions. The inverse of Θ is the mapping $\Phi: G \to (G/H)\times H$ given by $\Phi(x) = (\pi(x),\ (\Gamma(\pi(x)))^{-1}\cdot x)$. Then Φ is continuous and clearly $\Phi(x) = (H,x)$ when $x\in H$, so Φ is a homeomorphism leaving H "fixed pointwise". Finally, if $\pi_H:(G/H)\times H \twoheadrightarrow H$ denotes the natural projection, then it is easily verified that $r(x) := \pi_H\circ\Phi(x) = (\Gamma(\pi(x)))^{-1}\cdot x$ defines a continuous retraction from G onto H:

$$
\begin{tikzcd}
(G/H)\times H \arrow[r, "\Theta"] \arrow[d, "\mathrm{id}"'] & G \arrow[d, "\mathrm{id}"] \\
(G/H)\times H \arrow[d, "\pi_H"'] & G \arrow[l, "\Phi=\Theta^{\leftarrow}"'] \arrow[d, "\mathrm{id}"] \\
H & G \arrow[l, "r"]
\end{tikzcd}
$$

□

In some cases, the relation “is a ccs-subgroup of” is transitive.

Lemma 9. *Let G be a topological group and assume that H and K are closed subgroups of G, with H normal in G and contained in K. Then the following properties hold*:

(a) *If H is a ccs-subgroup of G and K/H is a ccs-subgroup of G/H, then K is a ccs-subgroup of G.*

(b) *If H is a ccs-subgroup of K and K is a ccs-subgroup of G, then K/H is a ccs-subgroup of G/H, and H is a ccs-subgroup of G.*

Proof. (a) follows from the fact that there exists a canonical bijection Φ of G/K onto $\frac{G/H}{K/H}$ (see [25, p. 111]). Moreover, it is easily checked that Φ is a homeomorphism between both coset spaces when they are equipped with the corresponding quotient topologies.

(b) By hypothesis, there exist continuous cross sections $\Gamma_1 : (K/H) \to K$ and $\Gamma_2 : (G/K) \to G$. Let

$$\pi : G \to G/H, \qquad \pi_K : G \to G/K$$

and

$$\pi_{K/H} : G/H \to \frac{G/H}{K/H} \cong G/K$$

be the canonical projections. Define

$$\Gamma_{K/H} : \frac{G/H}{K/H} \cong G/K \to G/H \quad \text{and} \quad \Gamma : G/H \to G,$$

by the rules

$$\begin{aligned} &\Gamma_{K/H}(gK) := (\pi \circ \Gamma_2)gK, \\ &\Gamma(gH) := \left[\Gamma_2 \circ \pi_{K/H}(gH)\right] \cdot \left[\Gamma_1\left(\left[(\Gamma_{K/H} \circ \pi_{K/H})(gH)\right]^{-1} \cdot gH\right)\right], \end{aligned}$$

for all $gK \in G/K$ and $gH \in G/H$, respectively. Both maps are continuous by definition, and it is seen without difficulty that each satisfies the requirements to be a continuous cross section for $\frac{G/H}{K/H}$, and G/H, respectively. $\square$

3. ccs-subgroups in the #-topology

Remarks 10.

(a) It is immediate from the definition that every homomorphism between #-groups is continuous. Accordingly for arbitrary groups G and H the topological groups $(G \times H)^{\#}$ and $G^{\#} \times H^{\#}$ coincide; we use the symbols $(G \times H)^{\#}$ and $G^{\#} \times H^{\#}$ interchangeably.

(b) It is easy to see [8, Section 2] that every subgroup H of a group $G^{\#}$ is closed in $G^{\#}$ and inherits the topology of $H^{\#}$. Thus in our context the three symbols $G^{\#}/H, G^{\#}/H^{\#}$, and $(G/H)^{\#}$ have identical import. Throughout this paper we use the third of these.

(c) If H is a closed topological direct summand of a topological group G in the sense that $G = (G/H) \times H$ both algebraically and topologically, then H is a ccs-subgroup of G. (The function $\Gamma : G/H \to G$ given by $\Gamma(x+H) := (x+H, 0)$ witnesses this.) It then follows from (a) above that if H is an (algebraic) direct summand, then $H^\#$ is a ccs-subgroup of $G^\#$. This yields the following result.

Lemma 11. *If H is a subgroup of G and G/H is a free group, then $H^\#$ is ccs-subgroup of $G^\#$.*

Proof. It is well known (cf. [22] or [16, 14.4]) that H is a direct summand of G. □

Lemma 12. *If H is a subgroup of G such that G/H is finitely generated, then $H^\#$ is a ccs-subgroup of $G^\#$.*

Proof. Write $G/H = F \times T$ with F a free group and T the torsion part of G/H [20, A.27]. Since $F \times \{0\}$ is free, there is a (necessarily continuous) homomorphism $f_0 : (F \times \{0\})^\# \to G^\#$ such that $\pi \circ f_0$ is the identity mapping on $F \times \{0\}$. Now for $t \in T$ note $F \times \{t\} = (F \times \{0\}) + (0, t)$, choose $x_t \in G$ such that $\pi(x_t) = (0, t)$, and define $f_t : F \times \{t\} \to G$ by $f_t(a, t) := f_0(a, 0) + x_t$. Then the function

$$\Gamma := \bigcup_{t \in T} f_t : (F \times T)^\# \to G^\#,$$

being continuous on each clopen coset $F \times \{t\}$, is a continuous cross section for G/H. □

The next two results are direct consequences from those in the previous section. The first one shows that among Abelian groups with the #-topology, the relation "is a ccs-subgroup of" is transitive.

Corollary 13.

(a) *Let $H^\#$ be a ccs-subgroup of $G^\#$, and let K be a subgroup of G such that $H \subseteq K$ and $(K/H)^\#$ is a ccs-subgroup of $(G/H)^\#$. Then $K^\#$ is a ccs-subgroup of $G^\#$.*

(b) *Let $H^\#$ be a ccs-subgroup of $K^\#$, and let $K^\#$ be a ccs-subgroup of $G^\#$. Then $(K/H)^\#$ is a ccs-subgroup of $(G/H)^\#$, and $H^\#$ is a ccs-subgroup of $G^\#$.*

Proof. It is a direct consequence of Lemma 9 and the topological isomorphism $(G^\#/H^\#) \cong (G/H)^\#$. □

The following result gives the connection between van Douwen's retract problem (Question 3 above) and the existence of continuous cross sections.

Corollary 14. *Let $H^\#$ be a ccs-subgroup of $G^\#$. Then*

(a) *there is a homeomorphism Φ of $G^\#$ onto $(G/H)^\# \times H^\#$ which restricted to H takes $h \mapsto (0, h)$; and*

(b) *$H^\#$ is a retract of $G^\#$.*

Proof. It is a direct consequence of Theorem 8. □

Definition 15.

(a) A topological group H is an *absolute ccs-group* for the class # if $H \in$ # and H is a ccs-subgroup of every #-group containing H as a closed subgroup.

(b) A topological group H is an *absolute retract for the class* # if $H \in$ # and H is a retract of every #-group containing H as a closed subgroup.

Notation 16.

(a) ACCS(#) is the class of absolute ccs-groups for the class #.

(b) AR(#) is the class of absolute retracts for the class #.

According to the conventions just introduced, Corollary 14(b) yields the following statement. This is perhaps one of the principal positive results of this paper, motivating our inquiry into the breadth (the extent) of the class ACCS(#).

Theorem 17. $\mathrm{ACCS}(\#) \subseteq \mathrm{AR}(\#)$.

Remark 18. In what follows, when a group H satisfies $H^{\#} \in \mathrm{ACCS}(\#)$ we sometimes say for simplicity that H is a *ccs-group*.

A divisible group D is algebraically a direct summand of any enveloping group, hence by Remark 10(c) satisfies $D^{\#} \in \mathrm{ACCS}(\#)$. The job of determining whether a given group H is a ccs-group is then simplified, as follows.

Theorem 19. *Let H be a group. Then*

(a) $H^{\#} \in \mathrm{ACCS}(\#)$ *if and only if* $H^{\#}$ *is a ccs-subgroup of its divisible hull* $(\mathrm{div}(H))^{\#}$; *and*

(b) $H^{\#} \in \mathrm{AR}(\#)$ *if and only if* $H^{\#}$ *is a retract of its divisible hull* $(\mathrm{div}(H))^{\#}$.

Proof. Necessity is obvious in each case. For the sufficiency, let G be a group containing H as a subgroup, and consider the inclusions $H \subseteq \mathrm{div}(H)$ and $\mathrm{div}(H) \subseteq \mathrm{div}(G)$. In (a) these are ccs inclusions, so by Corollary 13(b) there is a continuous cross section

$$\Gamma' : \big((\mathrm{div}(G))/H\big)^{\#} \to \big(\mathrm{div}(G)\big)^{\#}.$$

Clearly $\Gamma'[G/H] \subseteq G$, so $\Gamma := \Gamma'|_{(G/H)}$ is a continuous cross section for $(G/H)^{\#}$ into $G^{\#}$. In (b) there is a retraction $r_1 : (\mathrm{div}(H))^{\#} \twoheadrightarrow H^{\#}$ and (since $\mathrm{div}(H)$ is a direct summand of $\mathrm{div}(G)$) a retraction $r_2 : (\mathrm{div}(G^{\#})) \twoheadrightarrow (\mathrm{div}(H))^{\#}$. Then $r := (r_1 \circ r_2)|_{G^{\#}} : G^{\#} \twoheadrightarrow H^{\#}$ is a retraction. □

We interpret Theorem 19 as a statement that the ccs property for groups is a topological variant of (algebraic) divisibility.

Corollary 20. *The classes* ACCS(#) *and* AR(#) *are closed under finite products.*

Proof. Given $H_k \in \mathrm{ACCS}(\#)$ (respectively, $H_k \in \mathrm{AR}(\#)$) for $0 \leqslant k \leqslant n$, set

$$H := \prod_k H_k, \quad D_k := \mathrm{div}(H_k), \quad \text{and} \quad D := \prod_k D_k = \mathrm{div}(H).$$

Then with $\Gamma_k : (D_k/H_k)^\# \to D_k^\#$ (respectively, $r_k : D_k \twoheadrightarrow H_k$) as hypothesized, the map

$$\Gamma := \prod_k \Gamma_k : (D/H)^\# \to D^\#$$

(respectively, $r := \prod_k r_k : D \twoheadrightarrow H$) is as required; here we have used Remark 10(a), and the fact that a function into a product is continuous if its composition with each projection is continuous. □

Corollary 21. *Every finite cyclic group is a ccs-group.*

Proof. Let H be cyclic with $|H| = n$; we realize H in the form

$$H = \{k/n\colon\ 0 \leqslant k < n\} \subseteq \mathrm{div}(H) \subseteq [0, 1)$$

with addition in the group $[0, 1)$ taken $\bmod 1$. Let $A := [0, \frac{1}{n}) \cap \mathrm{div}(H)$. Since $\pi : (\mathrm{div}(H))^\# \twoheadrightarrow (\mathrm{div}(H)/H)^\#$ is an open map and A is open in $(\mathrm{div}(H))^\#$, the function $\pi|_A : A \twoheadrightarrow (\mathrm{div}(H)/H)^\#$ is a homeomorphism. Then $\Gamma := (\pi|_A)^{-1}$ is a continuous cross section for $(\mathrm{div}(H)/H)^\#$, as required. □

Corollary 22. *Every finite group is a ccs-group.*

Proof. By [20, 25.9 or A.25], such a group is a finite product of finite cyclic groups. □

It is natural to ask whether we can remove the word *finite* in the last three corollaries. We address these questions in Remark 44 through Question 47; in Theorem 24; and in Theorem 35, respectively.

4. Finitely generated groups are ccs-groups

This section is devoted to proving the result in its title. Every finitely generated group is a product of finitely many cyclic groups (cf. [20, A.27]), hence by Corollaries 20 and 21 it is enough by Theorem 19 to show that $\mathbb{Z}^\#$ is a ccs-subgroup of $Q^\#$, where Q denotes the rational group with its discrete topology. We do this in Theorem 24. We finish the section with additional examples of ccs-groups.

For an arbitrary topological group G we denote by $\widehat{G}$ the set of continuous homomorphisms from G to $\mathbb{T}$. The *Bohr* topology of G is the weakest topology that makes the elements of $\widehat{G}$ continuous. We define G^+ to be the underlying group of G equipped with its Bohr topology. If G is a subgroup of a locally compact Abelian group, then G^+ is a Hausdorff totally bounded group; we refer the reader interested in this subject to the extensive bibliography in our article [4]. For present purposes, we specialize to the topological groups $\mathbb{R}$, $\mathbb{Q}$, and $\mathbb{Q}/\mathbb{Z}$. Of course when G is discrete we have $G^\# = G^+$. We notice

(a) $\widehat{\mathbb{Q}} = \widehat{\mathbb{R}}$,
(b) $\mathbb{Q}^+$ is a topological subgroup of $\mathbb{R}^+$, and
(c) $\mathbb{Q}/\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})^+$;

indeed (b) is a consequence of (a), which follows from the fact that the elements of $\widehat{\mathbb{Q}}$ are uniformly continuous on $\mathbb{Q}$, and (c) is a particular case of the fact that every totally bounded topological group satisfies $G = G^+$ (cf. [7, 1.2]).

The expression *transversal set* was defined in Remark 5.

Now let $\phi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ be the canonical projection, and set $S := (-1/\sqrt{2}, 1 - 1\sqrt{2}) \cap \mathbb{Q}$. Then $\Phi := \phi|_S^{-1} : \mathbb{Q}/\mathbb{Z} \twoheadrightarrow S \subseteq \mathbb{Q}$ is a continuous cross section for $\mathbb{Q}/\mathbb{Z}$, so S is a transversal set for $\mathbb{Q}/\mathbb{Z}$; further, $\phi|_S$ is a homeomorphism of S onto $\mathbb{Q}/\mathbb{Z}$. We will show in Theorem 24 that Φ remains a continuous cross section for $(\mathbb{Q}/\mathbb{Z})^\#$; Lemma 23 relates to yet a third topological space, $(\mathbb{Q}/\mathbb{Z})^\#$.

The trivial function $x \mapsto x$ is a continuous bijection from $\mathbb{R}$ onto $\mathbb{R}^+$, hence is a homeomorphism on every compact subset of $\mathbb{R}$. Since S is contained in a compact subset of $\mathbb{R}$, it follows from (b) above that S inherits identical topologies from $\mathbb{Q}$ and from $\mathbb{Q}^+$.

Lemma 23. *Let $\phi^+ : \mathbb{Q}^+ \to \mathbb{Q}^+/\mathbb{Z} = (\mathbb{Q}/\mathbb{Z})^+$ be the canonical projection. Then $\Phi : \mathbb{Q}^+/\mathbb{Z} \to S \subset \mathbb{Q}^+$ is a continuous cross section for $\mathbb{Q}^+/\mathbb{Z}$, with transversal set S.*

Proof. Follows from the diagram

$$\begin{tikzcd}
\mathbb{Q} \arrow[r, equal] & \mathbb{Q} \arrow[r] \arrow[d, "\phi"'] & \mathbb{Q}^+ \arrow[r, equal] \arrow[d, "\phi^+"] & \mathbb{Q}^+ \\
S \arrow[u, "\subseteq"] \arrow[r, "\phi|_S"] & \mathbb{Q}/\mathbb{Z} \arrow[r, equal] & \mathbb{Q}/\mathbb{Z} \arrow[r, "\Phi"] & S \arrow[u, "\subseteq"']
\end{tikzcd} \qquad \square$$

We next use the fact that, if G is a topological group, $\widehat{G}$ becomes a group with operation defined pointwise:

$$\lambda_1, \lambda_2 \in \widehat{G}, x \in G \Rightarrow (\lambda_1 + \lambda_2)(x) := \lambda_1(x)\lambda_2(x).$$

Theorem 24. *$\mathbb{Z}^\#$ is a ccs-subgroup of $Q^\#$.*

Proof. Let $\pi : Q^\# \to Q^\#/\mathbb{Z} = (Q/\mathbb{Z})^\#$ and $\phi : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ be the canonical quotient homomorphisms. As sets, $\phi = \pi$. Let Φ be as in Lemma 23, and set $\Gamma := \Phi$. To prove that $\Gamma : (Q/\mathbb{Z})^\# \to Q^\#$ is a continuous cross section for $(Q/\mathbb{Z})^\#$, it is enough to show that $\chi \circ \Gamma : (Q/\mathbb{Z})^\# \to \mathbb{T}$ is continuous for all $\chi \in \widehat{Q}$:

$$(Q/\mathbb{Z})^\# \xrightarrow{\Gamma} Q^\# \xrightarrow{\chi} \mathbb{T}.$$

Now, let $\widehat{\pi_0} : \widehat{Q/\mathbb{Z}} \to \widehat{Q}$ be the dual mapping associated to the canonical projection π_0 of Q onto $Q/\mathbb{Z}$. That is, $\widehat{\pi_0}$ is defined by $\widehat{\pi_0}(\gamma)(q) := \gamma(\pi_0(q))$ for all $q \in Q$ and $\gamma \in \widehat{Q/\mathbb{Z}}$:

$$Q \xrightarrow{\pi_0} Q/\mathbb{Z} \xrightarrow{\gamma} \mathbb{T}.$$

Similarly, let $\hat{i} : \widehat{\mathbb{R}} \to \widehat{Q}$ be the dual mapping associated to the injection $i : Q \to \mathbb{R}$, where $\mathbb{R}$ is endowed with its standard topology. That is, $\hat{i}$ is defined by $\hat{i}(\gamma)(q) := \gamma(i(q))$ for all $q \in Q$ and $\gamma \in \widehat{\mathbb{R}}$:

$$Q \xrightarrow{i} \mathbb{R} \xrightarrow{\gamma} \mathbb{T}.$$

Thus we have

$$\widehat{Q/\mathbb{Z}} \xrightarrow{\widehat{\pi_0}} \widehat{Q} \xleftarrow{\hat{i}} \widehat{\mathbb{R}}.$$

It is known (cf. [10, Exercise 3.8.19(c)]) that the group $\widehat{Q}$ is generated by its subgroups $\widehat{\pi_0}[\widehat{Q/\mathbb{Z}}]$ and $\hat{i}[\widehat{\mathbb{R}}]$. The referee has offered a direct proof, as follows: Let $\chi \in \widehat{Q}$. If we define $\delta : Q \to \mathbb{T}$ by $\delta(x) := (\chi(1))^x$ then it is clear that $\delta \in \hat{i}[\widehat{\mathbb{R}}]$. Now define $\gamma : Q \to \mathbb{T}$ by $\gamma(x) := \chi(x) \cdot \delta(x)^{-1}$. Then $\gamma(1) = 1$ which implies that $\ker \gamma \supset \mathbb{Z}$. Thus there is $\widetilde{\gamma} \in \widehat{Q/\mathbb{Z}}$ such that $\widehat{\pi_0}[\widetilde{\gamma}] = \gamma$, i.e., $\gamma \in \widehat{\pi_0}[\widehat{Q/\mathbb{Z}}]$. Notice finally, completing the referee's proof, that $\chi = \gamma \cdot \delta$. Thus to check the continuity of $\chi \circ \Gamma$ (χ arbitrary) it is enough to consider separately the cases $\chi \in \widehat{\pi_0}[\widehat{Q/\mathbb{Z}}]$, $\chi \in \hat{i}[\widehat{\mathbb{R}}]$.

First let $\chi = \widehat{\pi_0}(\gamma) \in \widehat{\pi_0}[\widehat{Q/\mathbb{Z}}]$. For $\zeta \in (Q/\mathbb{Z})^\#$ we have

$$(\chi \circ \Gamma)(\zeta) = \big(\widehat{\pi_0}(\gamma)\big)\big(\Gamma(\zeta)\big) = \gamma\big(\pi_0(\Gamma(\zeta))\big) = \gamma(\zeta),$$

because $\pi_0 = \pi = p$ and $\Phi = \Gamma$ as sets, and Φ is a continuous cross section for $\mathbb{Q}/\mathbb{Z}$. Since $\gamma : (Q/\mathbb{Z})^\# \to \mathbb{T}$ is continuous by the definition of $(Q/\mathbb{Z})^\#$, and since $\chi \circ \Gamma = \gamma$, the function $\chi \circ \Gamma$ is continuous in this case.

Next let $\chi = \hat{i}(t) \in \hat{i}[\widehat{\mathbb{R}}]$ with $t \in \widehat{\mathbb{R}}$. For $\zeta \in (Q/\mathbb{Z})^\#$ we have

$$(\chi \circ \Gamma)(\zeta) = \hat{i}(t)\big(\Gamma(\zeta)\big).$$

Given that t is continuous on $\mathbb{R}^+$, and that $\Gamma = \Phi$ is also continuous for $\mathbb{Q}^+$ (or $\mathbb{R}^+$) by Lemma 23, we deduce that $\chi \circ \Gamma$ is continuous since always the # topologies are finer than the $+$ topologies:

$$\begin{array}{ccc}
(Q/\mathbb{Z})^\# & \xrightarrow{\mathrm{id}} & \mathbb{Q}/\mathbb{Z} \\
\Gamma \downarrow & & \downarrow \Phi \\
Q^\# & \xrightarrow{\mathrm{id}} & \mathbb{Q}^+ \\
\hat{i}(t) \downarrow & & \downarrow t|_{\mathbb{Q}^+} \\
\mathbb{T} & \longrightarrow & \mathbb{T}
\end{array}$$

The proof is complete. $\square$

Remark 25. That the subgroups $\widehat{\pi_0}[\widehat{Q/\mathbb{Z}}]$ and $\hat{i}[\widehat{\mathbb{R}}]$ generate the full group $\widehat{Q}$ is implicitly stated in [20, 25.4, 25.3, 25.2 and 10.12]. We are grateful to Dikran Dikranjan for this information. For more on the subject, the reader is invited to read the first two paragraphs of the Notes to Section 25 in [20].

Corollary 26.

(a) *$Q^\#$ is homeomorphic to $(Q/\mathbb{Z})^\# \times \mathbb{Z}^\#$.*
(b) *$\mathbb{Z}^\#$ is a retract of $Q^\#$.*

Proof. From Corollary 14 and Theorem 24. □

As mentioned in the introduction, it was noticed in [26, 6.33 and 6.36] that if G contains a subgroup H of index $n \in \mathbb{N}$ such that $G^\#$ and $H^\#$ are homeomorphic, then $G^\#$ is homeomorphic to $G^\# \times \mathbb{Z}_n$. In particular, $\mathbb{Z}^\#$ and $\mathbb{Z}^\# \times \mathbb{Z}_n$ are homeomorphic for each $n \in \mathbb{N}$. For groups G and H, Kunen [23] writes $G \sim H$ if there are subgroups G_1 of G, and H_1 of H, each of finite index, such that G_1 and H_1 are isomorphic, he notes that if $G \sim H$, then $G^\#$ and $H^\#$ are homeomorphic, and he asked whether the converse follows. Corollary 26(a) shows that the answer is "not always".

Corollary 27. *Every finitely generated group H satisfies $H^\# \in \mathrm{ACCS}(\#)$.*

Proof. From Theorems 19 and 24. □

For emphasis we state this consequence of Theorem 19 and Corollary 27.

Corollary 28. *Let H be a finitely generated group, and suppose G contains H as a subgroup. Then*

(a) *$G^\#$ is homeomorphic to $(G/H)^\# \times H^\#$, and*
(b) *$H^\#$ is a retract of $G^\#$.*

We note next that some ccs-groups are neither divisible nor finitely generated.

Theorem 29. *Let $\mathbb{D}_p$ denote the set of rational numbers of the form m/p^k, where $m \in \mathbb{Z}$, $p \in \mathbb{P}$ (the set of prime numbers) and $k \in \mathbb{N}$. Then every group of the form $\prod_{i=1}^n \mathbb{D}_{p_i}$ is a ccs-group.*

Proof. Being isomorphic to $\mathbb{Z}(p^\infty)$, the group $\mathbb{D}_p/\mathbb{Z}$ is divisible and therefore a ccs-subgroup of $Q/\mathbb{Z}$. Set $G := Q$, $H := \mathbb{Z}$ and $K := \mathbb{D}_p$, and apply Corollary 13(a) to see that $\mathbb{D}_p$ is a ccs-subgroup of (its divisible hull) Q. Now the result follows from Theorem 19(a) and Corollary 20. □

These results show that there are nondivisible groups (for example, finitely generated groups), which behave as injectives within the class of topological groups equipped with their maximal totally bounded topologies and continuous homomorphisms, in the following sense.

Remark 30. Let us say that a topological group G is an *absolute extensor for the class* #, and write $G \in \mathrm{AE}(\#)$, if (a) $G \in \#$ and (b) for every group $H^\#$ and every (necessarily closed) subgroup H_0 of $H^\#$, every (continuous) homomorphism $h : H_0 \to G$ extends to a continuous function $\tilde{h}: H^{\#} \to G$. There is in the literature a rich and extensive collection of theorems of the form "every AR($\mathbb{Q}$) is an AE($\mathbb{Q}$)" (for suitable classes $\mathbb{Q}$ of topological spaces). Accordingly, the following result is no surprise.

Theorem 31. AR(#) = AE(#).

Proof. Surely every topological group $G \in$ AE(#) satisfies $G \in$ AR(#): Given G a (closed) subgroup of $K \in$ #, the identity function $h := \mathrm{id}_G$ extends to a continuous function $\tilde{h}: K^{\#} \twoheadrightarrow G^{\#} = G$, i.e., to a retract. For the inclusion $\subseteq$, let $h: H_0 \to G \in$ AR(#) with h, H and H_0 as in Remark 30. It is well known (see, for example, [1,16, 21.1] or [20, A.7]) that h extends to a (necessarily continuous) homomorphism $\bar{h}: H^{\#} \to (\mathrm{div}(G))^{\#}$. Then let $r: (\mathrm{div}(G))^{\#} \twoheadrightarrow G^{\#} = G$ be a continuous retraction and take $\tilde{h} := r \circ \bar{h}$. □

5. An example of a group that is not a ccs-group

According to Corollary 14, to show that $H^{\#}$ is an absolute retract for the class # it is enough to show that $H^{\#}$ is a ccs-subgroup of every enveloping group $G^{\#} \in$ #. This leads one to wonder whether every group $H^{\#}$ is in ACCS(#). Unfortunately (for our purposes), the answer is "No". The goal in this section is to present examples of this phenomenon. Our construction relies heavily upon recent results given by Kunen [23]. For use below we record and adopt several notational devices from [23].

Let $q \in \mathbb{N}\backslash\{1\}$. Then $\mathbb{Z}_q$ denotes the ring $\{0, 1, 2, \ldots, q-1\}$, with addition and multiplication mod q.

The symbol $\mathbb{M}_q$ denotes the ω-dimensional $\mathbb{Z}_q$-module, i.e., $\mathbb{M}_q := \bigoplus_\omega \mathbb{Z}_q$. Notice that $\mathbb{M}_q$ can be viewed as the set of functions $f: \omega \to \mathbb{Z}_q$ such that $\mathrm{supp}(f) := \{n < \omega\colon f(n) \neq 0\}$ is a finite set; supp(f) is called *the support* of f. With this in mind, for $j < \omega$ we define $f_j \in \mathbb{M}_q$ as the characteristic function of the set $\{j\}$. Set $F := \{f_j\colon j < \omega\}$. Then F is an independent set in $\mathbb{M}_q$, and $\langle F \rangle = \mathbb{M}_q$. Note that in this case the notion of "independence" is group-theoretic [16, Section 16] or [20, A.10]; however, when q is a prime number, then vector space independence and group independence coincide.

For $n < \omega$, and $A \subseteq \omega$ we define $[A]^n := \{S \subset A\colon |S| = n\}$. The sets $[A]^{<n}$, $[A]^{\leqslant n}$, and $[A]^{<\omega}$ are similarly defined. We observe the following convention: If $s \in [\omega]^n$, we list the elements of s in *ascending order*, i.e., our notation $s = \{\alpha_0, \ldots, \alpha_{n-1}\}$ implies $\alpha_0 < \cdots < \alpha_{n-1}$. If $s \in [\omega]^{<\omega}$, let $f_s := \sum_{j \in s} f_j$, so $f_\emptyset = 0$. We identify $[\omega]^{<\omega}$ with a subspace of $\mathbb{M}_q$ through the map $s \mapsto f_s$. For example, $[\omega]^0 = \{f_\emptyset\}$, and $[\omega]^1 = F$.

Since every homomorphism from $\mathbb{M}_q$ to $\mathbb{T}$ takes $\mathbb{M}_q$ into $\mathbb{Z}_q$, the topology of $\mathbb{M}_q^{\#}$ may be characterized as the weakest topology that makes the elements of $\mathrm{Hom}(\mathbb{M}_q, \mathbb{Z}_q)$ continuous.

A *n-ary sequence* X (*indexed by* B) *from* $\mathbb{M}_q$ is a function $X: [B]^n \to \mathbb{M}_q$, where $B \subseteq \omega$; n is the *arity* of X. We identify X with its range. It is said to be *independent* if (the range of) X is an independent subset of $\mathbb{M}_q$. If Y is another n-ary sequence also indexed by B, and $c \in \mathbb{Z}_q$, then we define $X + Y: [B]^n \to \mathbb{M}_q$ and $cX: [B]^n \to \mathbb{M}_q$

by $(X+Y)(s) := X(s)+Y(s)$ and $(cX)(s) := cX(s)$. Notice that the sum and (scalar) product in the right hand side of the above two definitions are taken in (the $\mathbb{Z}_q$-module) $\mathbb{M}_q$. A *system* of *sequences* is a finite collection of sequences, indexed by (the same) B, not necessarily of the same arity. If $\chi = \{X_i\colon\ i<k\}$ is one such system, and n_i is the arity of X_i, then we say that *the system χ is independent* if $\{X_i(s)\colon\ i<k,\ s\in[B]^{n_i}\}$ is independent. As above, if $\chi=\{X_i\colon\ i<k\}$ and $c\in\mathbb{Z}_q$, then we define $c\chi := \{cX_i\colon\ i<k\}$.

If X is a n-ary sequence indexed by B and $A\subseteq B$, then we define

$$X\restriction A := \{X(s)\colon\ s\in[A]^n\}.$$

If χ is the system of sequences $\{X_i\colon\ i<k\}$ indexed by B and $A\subseteq B$, we similarly define $\chi\restriction A := \{X_i\restriction A\colon\ i<k\}$.

Finally assume that W is a m-ary sequence, X is a n-ary sequence, and both are indexed by B. We say that X is a *simple derived sequence from W* if $n\geqslant m$, and for some $i_0<\cdots<i_{m-1}<n$, we have

$$X\bigl(\{\alpha_0,\alpha_1,\ldots,\alpha_{n-1}\}\bigr) = W\bigl(\{\alpha_{i_0},\alpha_{i_1},\ldots,\alpha_{i_{m-1}}\}\bigr)$$

for all $\{\alpha_0,\alpha_1,\ldots,\alpha_{n-1}\}\in[B]^n$. Then if $\mathcal{W}=\{W_i\colon\ i<k\}$ is a system of sequences, we say that X is a *derived sequence* from $\mathcal{W}$ if $X=\sum_{\ell<L} c_\ell Y_\ell$ for some $L<\omega$, scalars $c_\ell\in\mathbb{Z}_q$ and sequences Y_ℓ, for $\ell<L$, where each Y_ℓ is a simple derived sequence from some W_i. Notice that we can assume each $c_\ell=1$, writing $Y'_\ell := c_\ell Y_\ell$ if necessary.

In all that follows we take for p a fixed prime and we set $q:=p^2$. Notice that $\mathbb{Z}_q$ contains a copy of $\mathbb{Z}_p$, namely the kernel of the map $x\mapsto px$. Similarly we see that $\mathbb{M}_q$ contains a copy of $\mathbb{M}_p$. It is elementary to notice that $\mathbb{M}_q/\mathbb{M}_p$ is isomorphic to $\mathbb{M}_p$. Hence, the natural map $\pi:\mathbb{M}_q\to\mathbb{M}_q/\mathbb{M}_p$ and the map $p:\mathbb{M}_q\to\mathbb{M}_p$ defined by $p(x):=px$, are equal.

If G is a group and $x\in G$, then the order of x is denoted by $o(x)$. Three preliminary results are needed for our main example. The first is a lemma.

Lemma 32. *Let X be a sequence in $\mathbb{M}_q$ such that the sequence $pX=\{px\colon\ x\in X\}$ is independent as a subset of $\mathbb{M}_p$. Then*

(a) *X is independent in $\mathbb{M}_q$.*

If in addition $o(x)=q$ for all $x\in X$, then

(b) *if $y\in\langle X\rangle$ is such that $o(y)=p$, then there exists $x\in\langle X\rangle$ with $px=y$; and*

(c) *for every homomorphism $\phi:H\to\langle X\rangle$ with H a subgroup of $\mathbb{M}_q$ and every $v\in\mathbb{M}_q\backslash H$, ϕ extends to a group homomorphism $\tilde\phi:\langle H\cup\{v\}\rangle\to\langle X\rangle$.*

Proof. (a) This is immediate.

(b) We have that $y=m_1x_1+m_2x_2+\cdots+m_nx_n$ with $x_i\in X$, $1\leqslant i\leqslant n$. Since y has order p and X is independent by (a), it follows that $m_i=pk_i$, $1\leqslant i\leqslant q$; therefore $y=pk_1x_1+pk_2x_2+\cdots+pk_nx_n$. Take $x:=k_1x_1+k_2x_2+\cdots+k_nx_n$.

(c) If $pv\notin H$, we define $\tilde\phi(v):=0$. If otherwise $pv\in H$, then $y:=\phi(pv)\in\langle X\rangle$. It is clear that $o(y)=p$, so by (b) there is $x\in\langle X\rangle$ such that $px=y$. We define $\tilde\phi(v):=x$ and extend $\tilde\phi$ over $\langle H\cup\{v\}\rangle$ by linearity. $\square$

For any set X, the diagonal $\Delta = \Delta_X$ is defined by $\Delta := \{(x, x)\colon x \in X\}$.

Lemma 33. *Let X be an n-ary sequence indexed by ω from $\mathbb{M}_q$. If each element in X has order q and every pair $(x, y) \in X^2 \setminus \Delta$ satisfies $x - y \notin \mathbb{M}_p$, then there exists an infinite $A \subseteq \omega$ and an independent system $\mathcal{W}$ such that $p\mathcal{W}$ is independent considered as a system in $\mathbb{M}_p$, all elements of the sequences in $\mathcal{W}$ have order q, and $X \restriction A$ is a derived sequence from $\mathcal{W} \restriction A$.*

Proof. Take the sequence pX and consider it as an n-ary sequence from $\mathbb{M}_p$. Apply Theorem 3.4 of [23] to obtain an independent system $\mathcal{V}$ of $\mathbb{M}_p$ and an infinite subset $B \subseteq \omega$ such that $pX \restriction B$ is a derived sequence from $\mathcal{V} \restriction B$. If $\mathcal{V} = \{V_i\colon i < k\}$, for every $i < k$ and $v \in V_i \restriction B$ we take $y_v \in \mathbb{M}_q$ such that $py_v = v$, and we define $Y_i := \{y_v\colon v \in V_i \restriction B\}$ and $\mathcal{Y} := \{Y_i\colon i < k\}$. By Lemma 32(a), $\mathcal{Y}$ is an independent system indexed by B from $\mathbb{M}_q$. Defining Y to be the n-ary sequence indexed by B from $\mathbb{M}_q$ which is derived from $\mathcal{Y}$ in exactly the same way that $pX \restriction B$ is derived from $\mathcal{V} \restriction B$, we see that all the members of Y have order q, and that $pX = pY$. Since every $(x, y) \in (X \restriction B)^2$ with $x \neq y$ satisfies $px \neq py$, it follows that there exists an n-ary sequence Z such that $X(s) = Y(s) + Z(s)$ for every $s \in [B]^n$, all of whose elements have order p. That is, $Z = (X \restriction B) + (p^2 - 1)Y$ is a sequence indexed by B from $\mathbb{M}_p \subset \mathbb{M}_q$.

Now just as in [23], it is convenient at this stage to assume that B belongs to a Ramsey ultrafilter Ψ. (This entails no loss of generality, since such Ψ exists in the forcing extension of the universe which makes CH true by collapsing 2^ω with countable conditions; since the present theorem involves only countable objects, its truth in the forcing extension implies its truth in the real universe. See [21, Lemma 19.6] with $\lambda = \omega$ for a stronger statement.) Then with $B \in \Psi$ we may apply Lemma 3.3 of [23] to the system $\mathcal{V}$ and the sequence Z from $\mathbb{M}_p$, thus achieving a system $\mathcal{V}'$ and an infinite subset $A \subseteq B$ such that the system $\mathcal{V} \cup \mathcal{V}'$ is independent and $Z \restriction A$ is a derived sequence from $(\mathcal{V} \cup \mathcal{V})' \restriction A$. Now, repeating the procedure used in the above section, we can define a system $\mathcal{Y}'$ in $\mathbb{M}_q$ related to $\mathcal{V}'$ in the same way that $\mathcal{Y}$ is related to $\mathcal{V}$. Hence the system $\mathcal{W} := \mathcal{Y} \cup \mathcal{Y}'$ is independent, all of its sequences have only elements of order q, $p\mathcal{W}$ is independent as a system from $\mathbb{M}_p$, and the sequences $Y \restriction A$ and $Z \restriction A$ are derived from $\mathcal{W} \restriction A$. Thus the sequence $(Y + Z) \restriction A = X \restriction A$ is also derived from $\mathcal{W} \restriction A$ and the proof is complete. □

Our next result is a variation of Theorem 4.1 in [23]. We give only a sketch of the proof.

Theorem 34. *Let p be a prime number and suppose that k is a multiple of p but not of $q := p^2$. Consider the space $[\omega]^k \cup \{\emptyset\}$ as a subspace of $\mathbb{M}_p^\#$. Let $\Gamma : [\omega]^k \cup \{\emptyset\} \to \mathbb{M}_q^\#$ be a mapping such that*

(a) *$\Gamma(\emptyset) = 0$,*

(b) *$\Gamma(s)$ has order q for all $s \in [\omega]^k$, and*

(c) *every $(s_1, s_2) \in ([\omega]^k)^2 \setminus \Delta$ satisfies $\Gamma(s_1) - \Gamma(s_2) \notin \mathbb{M}_p \subset \mathbb{M}_q$. Then Γ is not continuous.*

Proof. The strategy is to show that the continuity of Γ would imply the existence of $A \in [\omega]^\omega$ such that $\Gamma[[A]^k] = \{0\}$, contradicting (c). We sketch the argument. Define the k-ary sequence X from $\mathbb{M}_q$ by $X(s) := \Gamma(s)$, apply Lemma 33 to find an independent system $\mathcal{W}$ and an infinite subset $A \subseteq \omega$ such that $p\mathcal{W}$ is independent in $\mathbb{M}_p$, $X \restriction A$ is derived from $\mathcal{W} \restriction A$, and if $\mathcal{W} = \{W_i\colon\ i < n\}$, then every W_i consists only of elements of order q. We assume accordingly that $X = \sum_{\ell<L} Y_\ell$ where every Y_ℓ is a derived sequence of (some) W_i. Since every element of X has order q, there is an index $\ell_0 \in \mathbb{Z}_L$ such that Y_{ℓ_0} contains elements of order q. Set $Y := Y_{\ell_0}$ and $W := W_{i_0}$, where Y_{ℓ_0} is derived from W_{i_0}. If we consider the identity mapping $\mathrm{id}\colon\langle W\rangle \to \langle W\rangle$, then, since $p\mathcal{W}$ is independent in $\mathbb{M}_p$, we can apply Lemma 32(c) and the Kuratowski–Zorn Lemma to extend id to a homomorphism $\widetilde{\mathrm{id}}\colon\mathbb{M}_q \to \langle W\rangle$ such that $\widetilde{\mathrm{id}}[\langle W_i\rangle] = \{0\}$ if $i_0 \neq i$. Now take $G := \widetilde{\mathrm{id}} \circ X$.

Denote the arity of W by r, and notice that $0 \leqslant r \leqslant k$. The proof is completed by analyzing the possible values of r. There is no problem if $r = 0$, and if $r \geqslant 1$, then one argues *mutatis mutandis* as in the last two paragraphs in the proof of Theorem 4.1 in [23]. □

Theorem 35. *The group $\mathbb{M}_p^\#$ is not a ccs-subgroup of $\mathbb{M}_q^\#$.*

Proof. Suppose that $\Gamma\colon(\mathbb{M}_q/\mathbb{M}_p)^\# \to \mathbb{M}_q^\#$ is a continuous cross section. The identification of $\mathbb{M}_q/\mathbb{M}_p$ with $\mathbb{M}_p$ yields a mapping $\Gamma\colon\mathbb{M}_p^\# \to \mathbb{M}_q^\#$ which, when restricted to $[\omega]^{q+p} \cup \{\emptyset\}$, satisfies all the hypothesis of Theorem 34 above with $k = q + p$. Thus Γ cannot be continuous. □

Remark 36. The construction just given furnishes non-ccs-subgroups in groups of arbitrary cardinality $\kappa > \omega$. Continuing earlier notation with p prime and $q := p^2$, let $G := \bigoplus_\kappa \mathbb{Z}_q$ and $H := \bigoplus_\kappa \mathbb{Z}_p$, choose $I \in [\kappa]^\omega$ and define

$$J := \kappa\backslash I, \quad G_I := \bigoplus_I \mathbb{Z}_q, \quad G_J := \bigoplus_J \mathbb{Z}_q,$$

$$H_I := \bigoplus_I \mathbb{Z}_p, \quad \text{and} \quad H_J := \bigoplus_J \mathbb{Z}_p.$$

Now suppose that

$$\Gamma'\colon\left(\frac{G}{H}\right)^\# = \left(\frac{G_I}{H_I}\right)^\# \times \left(\frac{G_J}{H_J}\right)^\# \to G^\# = G_I^\# \times G_J^\#$$

is a continuous cross section, and define the injection map

$$\mathrm{inj}\colon\left(\frac{G_I}{H_I}\right)^\# \to \left(\frac{G_I}{H_I}\right)^\# \times \left(\frac{G_J}{H_J}\right)^\#$$

by

$$\mathrm{inj}(g + H_I) = (g + H_I, 0_J + H_J) \in \left(\frac{G_I}{H_I}\right)^\# \times \left(\frac{G_J}{H_J}\right)^\#.$$

Then with $p_I : G_I^\# \times G_J^\# \twoheadrightarrow G_I^\#$ the projection map define $\Gamma : (G_I/H_I)^\# \to G_I^\#$ by $\Gamma := p_I \circ \Gamma' \circ \mathrm{inj}$. Then Γ is continuous, and for $g + H_I \in G_I/H_I$ we have

$$(\Gamma' \circ \mathrm{inj})(g + H_I) \in (g + H_I) \times (0_J + H_J) \subseteq G_I^\# \times G_J^\#.$$

Thus $\Gamma(g + H_I) = p_I((\Gamma' \circ \mathrm{inj})(g + H_I)) \in g + H_I \subseteq G_I$, and with $\pi_I : G_I \to \frac{G_I}{H_I}$ the natural map we have $(\pi_I \circ \Gamma)(g + H_I) = g + H_I$. Thus Γ is a continuous cross section for $(G_I/H_I)^\#$, contrary to Theorem 35.

6. Final remarks and questions

Van Douwen's problem, Question 3 above, remains unsolved. We do not know if the pair $(G, H) = (\mathbb{M}_p, \mathbb{M}_q)$ can respond negatively:

Question 37. Is $\mathbb{M}_p^\#$ a retract of $\mathbb{M}_q^\#$?

We note that if such a retraction exists, then it cannot act linearly on each coset:

Theorem 38. *If H is a subgroup of G and $r : G^\# \to H^\#$ is a continuous retraction such that $r(x) - r(y) = x - y$ whenever $xH = yH$, then $H^\#$ is a ccs-subgroup of $G^\#$. Conversely, if $H^\#$ is a ccs-subgroup of $G^\#$, then there is a continuous retraction $r : G^\# \to H^\#$ such that $r(x) - r(y) = x - y$ whenever $xH = yH$.*

Proof. The first statement follows by defining $\Gamma : (G/H)^\# \to G^\#$ by $\Gamma(\pi(x)) := x - r(x)$. For the converse, follow the proof of Theorem 8. □

Testing the limits of applicability of the class ACCS(#) to Question 3, we have the following unsolved problem.

Question 39. Is the inclusion AR (#) $\subseteq$ ACCS(#) valid?

Of course, Question 39 may be posed locally, as follows.

Question 40. Suppose that H is a subgroup of G such that $H^\#$ is a retract of $G^\#$. Must $H^\#$ be a ccs-subgroup of $G^\#$?

Let us note that if a retraction existed, say in the case of $\mathbb{M}_4^\#$ and $\mathbb{M}_2^\#$, it cannot be an *ad hoc* function. In what follows we define the character $p_q \in \mathrm{Hom}(\mathbb{M}_q, \mathbb{Z}_q)$ by

$$p_q(x) := \sum_{n<\omega} x_n \bmod q.$$

Theorem 41. *Let $r : \mathbb{M}_4^\# \to \mathbb{M}_2^\#$ be any mapping such that, if $x \in \mathbb{M}_4$ and $p_4(x) = 0$, then*

(a) $r(x)_n = 1$ *when* $n \in \mathrm{supp}(x)$, *and*

(b) $r(x)_n = 0$ *when* $n \notin \operatorname{supp}(x)$.
Then r is not continuous.

The proof of Theorem 41 is postponed until after Lemma 42. As preamble, we set $S_q := \{1, \ldots, q-1\}$ if $q \in \mathbb{N}\backslash\{1\}$ and, given an ordered m-tuple $\vec{\zeta} = (\zeta_1, \ldots, \zeta_m) \in S_q^m$ with $m < \omega$, we define $A(\vec{\zeta})$ to be all points $x \in \mathbb{M}_q$ such that

(1) $|\operatorname{supp}(x)| = m$, and
(2) the m-many non-0 entries in the vector $x \in \mathbb{M}_q$ are the numbers ζ_k, $1 \leqslant k \leqslant m$, in the indicated order.

Restated for clarity: Given $\vec{\zeta}$ as above, we have

$$A(\vec{\zeta}) = \bigl\{x \in \mathbb{M}_q\colon \operatorname{supp}(x) = \{n_1 < n_2 < \cdots < n_m\},\ x_{n_k} = \zeta_k \text{ for } 1 \leqslant k \leqslant m\bigr\}.$$

Lemma 42. *Let* $\vec{\zeta} = (\zeta_1, \ldots, \zeta_m) \subseteq S_q^m$ *satisfy* $\sum_{j \in S_m} \zeta_j = 0 \bmod q$. *Then* $0 \in \operatorname{cl}_{\mathbb{M}_q^\#} A(\vec{\zeta})$.

Proof. A basic neighborhood U of 0 in $\mathbb{M}_q^\#$ has the form $U = \bigcap_{j=1}^N \chi_j^{-1}(0)$ with $\chi_j \in \operatorname{Hom}(\mathbb{M}_q, \mathbb{Z}_q)$. Note that for each $\chi \in \operatorname{Hom}(\mathbb{M}_q, \mathbb{Z}_q)$ and each infinite $I \subseteq \omega$, there is infinite $S_\chi \subseteq I$ such that $\chi|_{\{f_n\colon n \in S_\chi\}}$ is constant, i.e., such that if $n, l \in S_\chi$, then $\chi(f_n) = \chi(f_l)$. Repeating N-many times, we find a fixed infinite set $S \subseteq \omega$ such that each χ_j $(1 \leqslant j \leqslant N)$ is constant on $\{f_n\colon n \in S\}$; say $\chi_j(f_n) = \eta_j \in \mathbb{Z}_q$ for $1 \leqslant j \leqslant N$.

Now fix $n_1 < \cdots < n_m$ all in S and define $c \in \mathbb{M}_q$ by

$$c_n := \begin{cases} \zeta_k & \text{if } n = n_k, \\ 0 & \text{otherwise.} \end{cases}$$

Then $c \in A(\vec{\zeta})$. Notice that $c = \sum_{k=1}^m \zeta_k f_{n_k}$. Hence

$$\begin{aligned} \chi_j(c) &= \sum_{k \in S_m} \zeta_k \chi_j(f_{n_k}) \bmod q = \sum_{k \in S_m} \zeta_k \eta_j \bmod q \\ &= \eta_j \Biggl(\sum_{k \in S_m} \zeta_k\Biggr) \bmod q = \eta_j 0 = 0 \end{aligned}$$

for each j, $1 \leqslant j \leqslant N$. Thus $c \in U \cap A(\vec{\zeta})$. □

Proof of Theorem 41. Take, for example, $\vec{\zeta} := (\zeta_1, \zeta_2, \zeta_3) = (2, 1, 1)$ and set $A := A(\vec{\zeta})$. Notice that $2 + 1 + 1 = 0 \bmod 4$, so Lemma 42 applies with $q = 4$ to give $0 \in \operatorname{cl}_{\mathbb{M}_4^\#} A$. Clearly $p_2(r(x)) = 3$ for $x \in A$, but $p_2(0) = 0$. Thus r is not continuous. □

As the terms are commonly used in general topology, a homeomorph of a space $X \in \mathrm{AR}(\mathbb{Q})$ (for some class $\mathbb{Q}$) is *ipso facto* in $\mathrm{AR}(\mathbb{Q})$. Somewhat whimsically, we ask the corresponding question for the class #.

Question 43. Let $H^\# \in \mathrm{AR}(\#)$, let H' be a topological space homeomorphic to $H^\#$, and let H' be closed in (some) $G^\#$. Must H' be a retract of $G^\#$?

Remark 44. Finally we clarify and rephrase a question suggested at the end of Section 3. It is known that for $\{H_i\colon i \in I\}$ a set of nondegenerate groups with $|I| \geqslant \omega$ and with

$H := \prod_{i\in I} H_i$, the product topology on H (determined by the groups $H_i^{\#}$) is not the topology of $H^{\#}$ [8, 2.6] . That is, the relation $\prod_{i\in I}(H_i^{\#}) \in \#$ fails; thus, trivially, the classes ACCS(#) and AR(#) are not closed under the formation of (infinite) products. Concerning the class ACCS(#) we can make a (slightly) less transparent assertion:

Theorem 45. *There is a set of groups* $\{H_i\colon\ i \in I\}$, *with each* $H_i^{\#} \in \mathrm{ACCS}(\#)$, *such that* $(\prod_{i\in I} H_i)^{\#} \notin \mathrm{ACCS}(\#)$.

Proof. Taking (for example) $p = 2$, $q = 4$ and $\kappa = \mathfrak{c} = 2^{\omega}$, we have from Remark 36 that $(\bigoplus_{\mathfrak{c}} \mathbb{Z}_2)^{\#}$ is not a ccs-subgroup of $(\bigoplus_{\mathfrak{c}} \mathbb{Z}_4)^{\#}$; thus the relation $(\bigoplus_{\mathfrak{c}} \mathbb{Z}_2)^{\#} \in \mathrm{ACCS}(\#)$ fails. Since algebraically $\bigoplus_{\mathfrak{c}} \mathbb{Z}_2 \cong (\mathbb{Z}_2)^{\omega}$ (cf. [16, p. 44]), we conclude $((\mathbb{Z}_2)^{\omega})^{\#} \notin \mathrm{ACCS}(\#)$. □

Remark 46. The example given in Theorem 45 is susceptible to substantial generalization, using the fact (cf. [6, 3.1]) that every finite (Abelian) group F and infinite cardinal κ satisfy $F^{\kappa} \cong \bigoplus_{2^{\kappa}} F$.

We do not know the answer to the following question.

Question 47. Let $\{H_i\colon\ i \in I\}$ be a set of groups, with each $H_i^{\#} \in \mathrm{AR}(\#)$. Define $H := \prod_{i\in I} H_i$. Is $H^{\#} \in \mathrm{AR}(\#)$?

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