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THE DENSITY NUCLEUS OF A TOPOLOGICAL GROUP

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Abstract. Given a topological group G (usually compact abelian), the authors study the poset $\mathcal{D} = \mathcal{D}(G)$ of dense subgroups of G and its impact on the algebraic structure of G. A key tool for this is the subgroup $\mathbf{den}(G) := \bigcap \mathcal{D}$.

Definition. For a cardinal $\kappa \geq 1$, a topological group is in the class $\mathcal{F}_f(\kappa)$ [$\mathcal{F}(\kappa)$; $\mathcal{F}_2(\kappa)$; $\mathcal{F}_{ad}(\kappa)$, respectively] if some family of κ-many dense subgroups of G is independent and freely generated [independent; pairwise independent; pairwise almost disjoint, respectively].

Let K be a compact abelian group. Then

1. $K \in \mathcal{F}_{ad}(2)$;
2. $K \in \mathcal{F}(2) \Leftrightarrow$ either $r(K) > 0$ or each leading Ulm-Kaplansky invariant of K is infinite;
3. there are $D_0, D_1 \in \mathcal{D}(K)$ such that $\mathbf{den}(K) = D_0 \cap D_1$;
4. $K \in \mathcal{F}(\kappa) \Leftrightarrow K \in \mathcal{F}_2(\kappa)$;
5. if K is torsion and $K \in \mathcal{F}(2)$, then $K \in \mathcal{F}(\kappa) \Leftrightarrow \kappa \leq$ each leading Ulm-Kaplansky invariant of K;
6. if $r(K) > 0$, then $K \in \mathcal{F}(\kappa) \Leftrightarrow \kappa \leq r(K)$; if, in addition, $r(K) \geq d(K)$, then $K \in \mathcal{F}_f(\kappa) \Leftrightarrow K \in \mathcal{F}(\kappa)$.

1. Introduction

1.1. Notation and terminology.

The least infinite cardinal is denoted ω, and $\mathfrak{c} := 2^\omega$.
For a set X and a cardinal κ, we write

$$[X]^\kappa := \{A \subseteq X : |A| = \kappa\}.$$

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The symbols $[X]^{\leq\kappa}$ and $[X]^{<\kappa}$ are defined analogously.

Two sets A_0 and A_1 are said to be *almost disjoint* if $|A_0 \cap A_1| < \omega$.

With rare exceptions noted explicitly *in situ*, we consider here only abelian groups. Emphasizing that convention, we use additive notation, denoting the identity of an abelian group G by $0 = 0_G$. (Occasionally, to indicate a "neutral" setting in which an abelian hypothesis is neither affirmed nor denied, we denote the identity element of a group G by $e = e_G$.) As a similar mnemonic device designed to be helpful to the reader, we use the symbol G for a general or generic (topological) group, K for a topological group known or assumed to be compact abelian.

We denote by $\mathbb{N}$ and $\mathbb{P}$ the sets of positive naturals and primes, respectively; by $\mathbb{Z}$ the integers; by $\mathbb{Q}$ the rationals; and by $\mathbb{R}$ the reals. The group of p-adic integers is denoted by $\mathbb{Z}_p$ $(p \in \mathbb{P})$, and $\mathbb{Z}(n)$ is the cyclic group of order $n > 1$. With each of these sets we associate, as needed, its usual algebraic and topological properties.

The subgroup generated by a subset X of a group G is denoted by $\langle X \rangle$.

The *free rank* of an abelian group G is denoted by $r(G)$; the *torsion subgroup* of G (i.e., the subgroup generated by the set of torsion elements of G) is denoted by $t(G)$, the p-torsion subgroup of G by $G(p)$. The group G is *free* if it has algebraically the form $\bigoplus_{i \in I} \mathbb{Z}_i$, with each $\mathbb{Z}_i$ a copy of the cyclic group $\mathbb{Z}$ ([32](§14)).

All topological spaces hypothesized in this paper are assumed to be Tychonoff spaces. Our topological groups are assumed to satisfy the T_0 separation axiom; as is well known (see for example [35, Theorem (8.4)]), our topological groups are, then, topological spaces.

As usual, for a space X the symbol $w(X)$ denotes the *weight* of X.

1.2. Historical remarks: Resolvability.

Edwin Hewitt [34] in 1943 initiated an extensive line of research, pursued over the years by many scholars, with this definition: A topological space is *resolvable* if it contains complementary dense subsets. (Hewitt showed, among other facts, that metrizable spaces without isolated points, and locally compact spaces without isolated points, are resolvable.) It was then natural for J. G. Ceder [1] to define, for an arbitrary cardinal $\kappa \geq 2$, the concept of κ-resolvability: A space $X = (X, \mathcal{T})$ is *κ-resolvable* if it admits a family of κ-many pairwise disjoint dense subsets, and it is *maximally resolvable* if it is κ-resolvable with

$$\kappa = \Delta(X) := \min\{|U| : \emptyset \neq U \in \mathcal{T}\}.$$

Building on Hewitt's work, Ceder [1] showed that infinite metric spaces $(X, \mathcal{T})$ with $w(X) \leq \Delta(X)$, and infinite locally compact spaces $(X, \mathcal{T})$ with $w(X) \leq \Delta(X)$, are maximally resolvable; later W. W. Comfort and Salvador García-Ferreira [8] established same conclusions without the hypothesis $w(X) \leq \Delta(X)$.

Generalizations of the resolvability concept were introduced and studied in subsequent decades by many workers, for example in [9], [10], [41], [43], [44]. In these works, broadly speaking, the requirement that the family of dense sets be pairwise disjoint was replaced by some weaker smallness condition—the sets should have pairwise intersections which are finite, for example, or countable, or nowhere dense, or of first category, or of measure zero when the underlying space is a measure space. In other instances the dense sets were required to be Borel, or to be topologically restricted (to be pseudocompact, for example). We refer the reader also to such works as [38], [11], [39], [12] for extensive relevant bibliographic references and for theorems relating to the existence of spaces, typically Tychonoff spaces, which satisfy certain prescribed resolvability properties but not others.

In the category of topological groups, the "correct" or natural question of resolvability type is not immediately obvious. Every subgroup of a group contains the identity, so it is pointless to seek pairwise disjoint subgroups of a given topological group. Broadly speaking, two fruitful lines of inquiry concerning resolvability in the group-theoretic context have developed in the literature.

(A) Given G, look for a proper, dense subgroup H of G. Then the coset decomposition witnesses that G is $|G/H|$-resolvable.
(B) Given G, look for a (possibly large) family $\{H_i : i \in I\}$ of subgroups of G such that $H_i \cap H_j = \{e_G\}$ (or perhaps $|H_i \cap H_j| < \omega$ for $i, j \in I, i \neq j$); consider, in addition, whether for some natural pre-assigned topological class $\mathbf{P}$ the dense subgroups H_i may be chosen so that $H_i \in \mathbf{P}$.

We note that while these two initiatives have differing thrusts—(A), after all, is strictly set-theoretic, while (B), in addition, sets algebraic requirements—they are hardly unrelated. Although (A) makes no immediate contribution to (B), the existence of two dense subgroups H_0 and H_1 such that $|H_0 \cap H_1| < \omega$ gives maximal resolvability of G: Indeed, if (say) $|H_0| < |G|$, then $|G/H_0| = |G|$ so the usual decomposition of G into H_0-cosets shows that G is $|G|$-resolvable, while if $|H_0| = |H_1| = |G|$, then $|G/H_0| = |H_1| = |G|$ and again G is maximally resolvable.

Concerning (A), M. Rajagopalan and H. Subrahmanian [46] showed by concrete example that there are infinite LCA groups with no proper dense subgroup. These examples justify and explain our choice in this paper to study dense subgroups of *compact* groups in the direction of line (B). Since every infinite compact group G has a dense subgroup H of maximal index, indeed with $|H| = d(G) \leq w(G) < 2^{w(G)} = |G|$ (see [36, Lemma 28.28(b)]), such a group is maximally resolvable.

For the reader's convenience we note in passing that V. I. Malykhin [42] showed in the axiom system [$\mathsf{ZFC} + \mathsf{MA}$] that the Boolean group $\oplus_\omega \{0,1\}$ admits no resolvable group topology, while in the converse direction, Comfort and Jan van Mill [15] showed that every abelian group not containing algebraically the group $\oplus_\omega \{0,1\}$ is resolvable in every nondiscrete group topology.

Malykhin has remarked to the second-listed co-author, and briefly in [44], that in some models of Zermelo-Fraenkel set theory (without AC), it follows from the work of A. G. El′kin [30] that every irresolvable topological group is discrete.

For other initiatives in this direction, showing the existence of large families of dense sets with small intersection when the underlying group is precompact (a.k.a. "totally bounded"), see [45] and other works by those same authors, and also [9] and [10].

Our work here lies in the direction of (B). Noting that a topological space with an isolated point cannot be resolvable, and that a point in a space X is isolated if and only if it belongs to every dense subset of X, we introduce this notation and terminology.

Definition 1.1. Let G be a topological group G. Then
(a) $\mathcal{D} = \mathcal{D}(G) := \{D : D \text{ is a dense subgroup of } G\}$, and
(b) $\mathbf{den}(G)$, the *density nucleus of* G, is the subgroup $\mathbf{den}(G) := \bigcap \mathcal{D}$.

So far as we can determine, no other workers heretofore have explicitly defined or studied the set-theoretic structure of the poset of dense subgroups of a given topological group. But many papers contain substantial, relevant results which deserve mention. We pass to a brief historical review.

1.3. Historical remarks: Dense subgroups.

The first researcher to address explicitly the issue of the existence of a proper dense pseudocompact subgroup in a given compact group K was Howard J. Wilcox; he answered the existence question affirmatively when there is κ such that $w(K) \leq 2^\kappa$ and $\kappa^\omega < 2^\kappa$ ([49, Theorem 1.15]). He showed also that not every compact abelian group of uncountable weight contains dense pseudocompact subgroups D_0 and D_1 such that $D_0 \cap D_1 = \{0\}$ ([49, §2.6]). The same paper shows (see 2.6) that with $\kappa > \omega$ and $K := ((\mathbb{Z}(2))^\kappa \times \mathbb{Z}(4)$, every dense pseudocompact subgroup of K contains $H := \{0\} \times \mathbb{Z}(2) = \{0\} \times 2\mathbb{Z}(4)$. (One easily sees a bit more, namely $H \subseteq \mathbf{den}(K)$, so in our notation we have an early and straightforward instance of a compact abelian group K such that $|\mathbf{den}(K)| > 1$.) In much the same vein, it is noted in [37, Example 1.7], taking $\kappa \geq \omega$ and $K := ((\mathbb{Z}(p)^\kappa)) \times \mathbb{Z}(q)$ $(p, q \in \mathbb{P}, p \neq q)$ that $\{0\} \times \mathbb{Z}(q) \subseteq \mathbf{den}(K)$, hence, again, $|\mathbf{den}(K)| > 1$.

In the interest of completeness, in Remark 2.9(b), we extrapolate the ideas of the preceding paragraph to something approximating maximal generality.

That every compact abelian group K with $w(K) > \omega$ has a proper dense pseudocompact subgroup H was apparently first proved in 1982 [20](§4.3). (The totally disconnected case had been settled in [23, Theorem 4.3].) Later, it was shown [2](4Cff.) that H may be chosen *ω-bounded* in the sense that every countable subset of H has compact closure in H.

The issue of the existence of many dense pseudocompact subgroups of a fixed compact group was approached by Comfort and Lewis C. Robertson [21](4.4(a)), who showed that pseudocompact (not necessarily abelian) groups of the form H^κ $(\kappa > \omega)$ admit 2^{2^κ}-many such subgroups. In the same direction Comfort and van Mill [16], responding in part to a question of Gerald Itzkowitz and Dmitri Shakhmatov [37], showed that a compact abelian group with weight $\kappa = \kappa^\omega$ has 2^{2^κ}-many dense ω-bounded subgroups. That every pseudocompact abelian group of uncountable weight has a proper dense pseudocompact subgroup was shown in [17]; see [18] for an incisive, unified treatment and for extensive references to preliminary and related results.

Of the several papers cited above and others mentioned collectively in their several bibliographies, the present work follows most closely the initiative of Itzkowitz and Shakhmatov [37], who enumerated a number of hypotheses on a topological group G sufficient to guarantee the existence of a large family $\{D_i : i \in I\}$ of dense pseudocompact subgroups of G whose intersections are small in one sense or another. In (approximately) increasing order of strength, they consider these possibilities:

(a) $D_i \neq D_j$ when $i, j \in I$, $i \neq j$;
(b) $D_i \cap D_j$ is not dense in G, when $i, j \in I$, $i \neq j$;
(c) $D_i \cap D_j$ is nowhere dense in G, when $i, j \in I$, $i \neq j$;
(d) $D_i \cap D_j = \{e\}$, when $i, j \in I$, $i \neq j$;
(e) $D_i \cap \langle \bigcup_{i \neq j \in I} D_j \rangle$ is nowhere dense in G, for $i \in I$; and
(f) $D_i \cap \langle \bigcup_{i \neq j \in I} D_j \rangle = \{e\}$, for $i \in I$.

Departing from the terminology favored in [37], we say that a family $\{D_i : i \in I\}$ of dense (not necessarily pseudocompact) subgroups as in (f) is an *independent* family. A family $\{D_i : i \in I\}$ of subgroups as in (d) is a *pairwise independent* family. As usual, a set $X \subseteq G$ is *independent* if the family $\{\langle x \rangle : x \in X\}$ of cyclic subgroups is independent in G.

Among many other results, the authors of [37] show that every compact group G which is connected or abelian, with $w(G) = \kappa > \omega$, admits a family of dense, pseudocompact $(D_i)_{i\in I}$ as in (e), with $|I| = 2^\kappa$. When G is both abelian and connected, the family may be chosen to be independent, but (as is shown by the examples mentioned above), that conclusion can fail if the connectivity hypothesis is omitted.

Weaker results of the same flavor, furnishing smaller (but independent) families of dense, pseudocompact subgroups under very limited and specialized hypotheses, had been given earlier, for example in [3, Theorem 3.1] and [13, Theorem 4.2].

For further results on dense subgroups see [3], [4], [5], [6], [13], [14], and [26].

We conclude this Introduction by listing four theorems from the literature. We will use these basic results as needed, routinely and casually. Proofs of theorems 1.2, 1.4, and 1.5 are given in [36, Lemma 28.28(c)], [13, Remark 2.17], and [14, Theorem 2.8], respectively, while Theorem 1.3 follows readily from the Baire category theorem; see also relation (2) of Remark 2.6(b).

Theorem 1.2. *Every infinite compact group G satisfies $|G| = 2^{w(G)}$.*

Theorem 1.3. *Every compact abelian torsion group is of bounded order.*

Theorem 1.4. *Every compact abelian group K such that $r(K) > 0$ satisfies $r(K) \geq \mathfrak{c}$.*

Theorem 1.5. *Every precompact abelian group such that $\omega < \kappa = w(G) = r(G)$ satisfies $G \in \mathcal{F}(\kappa)$; the witnessing dense subgroups may be chosen to be torsionfree.*

2. Fragmentation in Topological Groups

The foregoing material gives general background and motivation for our investigations. We continue with some material specific to our considerations.

Definition 2.1. Let $\kappa > 0$ be a cardinal. A topological group G is

(i) *κ-free-fragmentable* (in symbols: $G \in \mathcal{F}_f(\kappa)$) if G has an independent family $\{D_i : i \in I\}$ of dense free subgroups with $|I| = \kappa$;

(ii) *κ-fragmentable* (in symbols: $G \in \mathcal{F}(\kappa)$) if G has a independent family $\{D_i : i \in I\}$ of dense subgroups with $|I| = \kappa$;

(iii) *κ-pairwise-fragmentable* (in symbols: $G \in \mathcal{F}_2(\kappa)$) if G has a pairwise independent family $\{D_i : i \in I\}$ of dense subgroups with $|I| = \kappa$.

In addition, we write $G \in \mathcal{F}_{ad}(\kappa)$ if G has a family of κ-many pairwise almost disjoint dense subgroups.

Remark 2.2. (a) Every group is 1-fragmentable but, as section 1.3 and Theorem 2.7 show, there are many compact groups that fail to be 2-fragmentable.

(b) $\mathcal{F}_f(1)$ is the class of abelian topological groups with a free dense subgroup (see [26] for properties of this class).

(c) $\mathcal{F}(2) = \mathcal{F}_2(2)$.

(d) For $\kappa > 1$, we have the class-theoretic inclusions $\mathcal{F}_f(\kappa) \subseteq \mathcal{F}(\kappa) \subseteq \mathcal{F}_2(\kappa) \subseteq \mathcal{F}_{ad}(\kappa)$.

(e) Let G be an abelian topological group and let $\kappa > 1$ be a cardinal. Then

(e$_1$) if G is torsion free and $G \in \mathcal{F}_{ad}(\kappa)$, then $G \in \mathcal{F}_2(\kappa)$; and

(e$_2$) if G is free and $G \in \mathcal{F}(\kappa)$, then $G \in \mathcal{F}_f(\kappa)$.

(f) If $G \in \mathcal{F}_2(\kappa)$, then $\kappa \leq |G|$.

As our Abstract indicates, we (will) show in theorems C and D that for every compact abelian group K there is a biggest cardinal number κ such that $K \in \mathcal{F}(\kappa)$. For such groups, a similar phenomenon holds for the classes $\mathcal{F}_f(\kappa)$ and $\mathcal{F}_2(\kappa)$. This suggests the following notation, in parallel with the symbols $\mathcal{F}_f(\kappa)$, $\mathcal{F}(\kappa)$, $\mathcal{F}_2(\kappa)$, and $\mathcal{F}_{ad}(\kappa)$ already introduced. Of course, this definition and remarks dependent upon it have meaning only for those groups for which such biggest cardinals exist.

Definition 2.3. Let G be a topological abelian group and $\kappa \geq 2$. Then

(a) G has *free fragmentation number* κ (briefly, $\mathfrak{f}_f(G) = \kappa$), if $G \in \mathcal{F}_f(\kappa)$, but $G \notin \mathcal{F}_f(\kappa^+)$.

(b) G has *fragmentation number* κ (briefly, $\mathfrak{f}(G) = \kappa$), if $G \in \mathcal{F}(\kappa)$, but $G \notin \mathcal{F}(\kappa^+)$.

(c) G has *pairwise fragmentation number* κ (briefly, $\mathfrak{f}_2(G) = \kappa$), if $G \in \mathcal{F}_2(\kappa)$, but $G \notin \mathcal{F}_2(\kappa^+)$.

(d) G has *almost disjoint fragmentation number* κ (briefly, $\mathfrak{f}_{ad}(G) = \kappa$), if $G \in \mathcal{F}_{ad}(\kappa)$, but $G \notin \mathcal{F}_{ad}(\kappa^+)$.

We record in §4 several useful properties of the invariants just defined. As a corollary of our main results, we show that every compact abelian group K satisfies $\mathfrak{f}_2(K) = \mathfrak{f}(K)$ (see Corollary D). Thus, we lose nothing in choosing to focus attention more on the class $\mathcal{F}(\kappa)$ than on $\mathcal{F}_2(\kappa)$—these classes coincide in the context of compact abelian groups.

Definition 2.4. (a) G is *maximally fragmentable* if $\mathfrak{f}(G) = |G|$ (i.e., $G \in \mathcal{F}(|G|)$);

(b) G is *maximally free-fragmentable* if $\mathfrak{f}_f(G) = |G|$ (i.e., $G \in \mathcal{F}_f(|G|)$).

(c) G is *maximally almost disjoint-fragmentable* (briefly, *maximally ad-fragmentable*) if $\mathfrak{f}_{ad}(G) = |G|^\omega$ (i.e., $G \in \mathcal{F}_f(|G|)$).

Items (a) and (b) are motivated by item (f) of Remark 2.2, while (c) is motivated by item (b) of Theorem 5.2.

Remark 2.5. In view of Theorem 1.2, the theorem cited above from [37] may be stated as follows: *Every compact connected non-metrizable abelian group is maximally free-fragmentable—and indeed, the witnessing dense subgroups may be chosen pseudocompact.*

We shall see in what follows that certain simple algebraic conditions suffice to vitiate maximal fragmentability. For example, when a topological abelian group G is not a bounded torsion group, then necessarily $\mathfrak{f}(G) \leq \min\{|mG| : m \in \mathbb{Z}, mG \neq \{0\}\}$ (see Lemma 4.4), so if $\min\{|mG| : m \in \mathbb{Z}, mG \neq \{0\}\} < |G|$, then G cannot be maximally fragmentable. (In case G is compact, then $\min\{|mG| : m \in \mathbb{Z}, mG \neq \{0\}\} = r(G)$.) Similarly, when G is bounded torsion, then $\mathfrak{f}(G) \leq \min\ell_{UK}(G)$ $(= \min\{\ell^p_{UK}(G) : p \in R(G)\})$ (see Lemma 4.2 and Corollary 4.5), so G cannot be maximally fragmentable if $\min\ell_{UK}(G) < |G|$. (See Remark 2.6 for a definition of the notations $R(G)$, $\min\ell_{UK}(G)$, and $\ell^p_{UK}(G)$.) In the same vein, it is obvious that $\kappa \leq r(G)$ if $G \in \mathcal{F}_f(\kappa)$, so a compact abelian group with $r(G) < |G|$ cannot be maximally free-fragmentable.

2.1. The poset of dense subgroups.

Remark 2.6. Here, introducing important notation to be used repeatedly later, we recall some basic facts from [35, Theorem (A.3)] and [32, Theorem 8.4; §37].

(a) Each abelian torsion group G has algebraically the form $G = \bigoplus_{p\in\mathbb{P}} G(p)$, where $G(p)$ is a p-group. The *representation set* of G, denoted $R(G)$, is the set

$$R(G) := \{p \in \mathbb{P} : |G(p)| > 1\};$$

it is clear that the torsion group G is of bounded order if and only if $|R(G)| < \omega$ and each $G(p)$ $(p \in R(G))$ is of bounded order. For such G, each subgroup $G(p)$ $(p \in R(G))$ has algebraically the form

$$G(p) = \bigoplus_{1\leq k\leq n_p} \bigoplus_{\alpha_{k,p}} (\mathbb{Z}(p^k)), \tag{1}$$

(with $n_p < \omega$ for each prime p). The cardinal numbers $\{\alpha_{k,p} : 1 \leq k \leq n\}$ are the *Ulm-Kaplansky invariants* of $G(p)$; $\alpha_{n_p,p}$ is the *leading* p-invariant of G. For $G = \bigoplus_{p\in R(G)} G(p)$, we write $\ell^p_{UK}(G) = \ell_{UK}(G(p)) := \alpha_{n_p,p}$. For an arbitrary abelian torsion group of bounded order, we set $\min\ell_{UK}(G) := \min\{\ell^p_{UK}(G) : p \in R(G)\}$. If $G = G(p)$ is a p-group, we simply write n in place of n_p and α_k in place of $\alpha_{k,p}$ for all $1 \leq k \leq n = n_p$.

(b) The topological and algebraic structure of compact abelian torsion groups has been fully determined and well understood for many years (see for example [35, Theorem (25.9)]). Indeed, such a group K has topologically and algebraically the form $K = \bigoplus_{p\in R(K)} K(p)$ with $|R(K)| < \omega$ and with

$$K(p) = (\mathbb{Z}(p))^{\epsilon_1} \times (\mathbb{Z}(p^2))^{\epsilon_2} \times \ldots \times (\mathbb{Z}(p^n))^{\epsilon_{n_p}}. \tag{2}$$

(Note that Theorem 1.3 is immediate from that structure theorem. Note also that the topological structure of $K(p)$ determines the cardinal number ϵ_{n_p}, as $\epsilon_{n_p} = w(p^{n_p-1}K(p))$ when $\epsilon_{n_p} \geq \omega$, while $p^{n_p-1}K(p)$ is the (finite, discrete) group $(\mathbb{Z}(p))^{\epsilon_{n_p}}$ when $\epsilon_{n_p} < \omega$.)

Since every finite abelian group F algebraically satisfies $F^\kappa = \bigoplus_{2^\kappa} F$ for every infinite cardinal $\kappa > 1$ ([32, §8]), equation (1) takes algebraically the form

$$K(p) = \bigoplus_{1\leq k\leq n_p} \bigoplus_{\alpha_k} \mathbb{Z}(p^k), \tag{3}$$

where, for every p and for $1 \leq k \leq n_p$, the Ulm-Kaplansky invariant α_k of $K(p)$ is either equal to ϵ_k, when ϵ_k is finite, or $\alpha_k = 2^{\epsilon_k}$ when ϵ_k is infinite.

If an abelian p-group K has ϵ_n finite, then $\{0\} \neq p^{n-1}K \cong (\mathbb{Z}(p))^{\epsilon_n} \subseteq \mathbf{den}(K)$ (see Theorem 2.7).

Using the presentation of the compact torsion groups given in Remark 2.6(b), we discuss now when the subgroup $\mathbf{den}(G)$ of a (compact torsion) group G is non-trivial (clearly, a group G such that $\mathbf{den}(G) > 1$ does not belong to the class $\mathcal{F}(2)$).

Theorem 2.7. *Let K be a compact abelian group. Then every $m \in \mathbb{Z}$ such that $|mK| < \omega$ satisfies $mK \subseteq \mathbf{den}(K)$.*

Proof. For $D \in \mathcal{D}(K)$ the set mD is dense in the (finite) set mK, so we have $mK = mD \subseteq D$. □

Remark 2.8. We note in passing that with obvious minor modifications the proof of Theorem 2.7 suffices to prove a more general statement (not needed in this work): *If G is a topological group that for some $m \in \mathbb{Z}$ has an open central subgroup H of exponent m, then $x^m \in D$ for every $x \in G$ and every dense subgroup D of G.*

Theorem 2.7 has additional consequences.

Remark 2.9. (a) Let K be an infinite abelian compact torsion group. If $K \in \mathcal{F}(2)$, then every primary component of K is infinite. Indeed, assume that some primary component $K(p)$ of K is finite, so that $K = K(p) \times L$ for some (necessarily closed) subgroup L of K of finite exponent

m that is coprime with p. Then every dense subgroup D of K contains $K(p) = mK$, i.e., $mK \subseteq \mathbf{den}(K)$, according to Theorem 2.7. This yields many examples of infinite abelian compact torsion groups $K \notin \mathcal{F}(2)$; for example, $K = \mathbb{Z}(3) \times (\mathbb{Z}(2))^\omega$.

(b) Let $p \in \mathbb{P}$ and let $K = \mathbb{Z}(p^k) \times (\mathbb{Z}(p))^\omega$ with $1 < k < \omega$. Then $K \notin \mathcal{F}(2)$. Indeed, from Theorem 2.7 with $H = \{0\} \times (\mathbb{Z}(p))^\omega$, we have $\emptyset \neq pK = \mathbb{Z}(p^{k-1}) \subseteq \mathbf{den}(K)$.

This example can be substantially generalized. According to (2) in Remark 2.6(b), every compact abelian p-group K has necessarily the form

$$K = (\mathbb{Z}(p))^{\alpha_1} \times (\mathbb{Z}(p^2))^{\alpha_2} \times \ldots \times (\mathbb{Z}(p^n))^{\alpha_n}$$

with leading Ulm-Kaplansky invariant $\ell^p_{UK}(K) = 2^{\alpha_n}$. Then, as above, if $\alpha_n < \omega$, then $\{0\} \neq p^{n-1}K \cong (\mathbb{Z}(p))^{\alpha_n} \subseteq \mathbf{den}(K)$.

(c) With K and m as in Theorem 2.7, the subgroup $H := \{x \in K : mx = 0\}$ of K is open in *every* compact Hausdorff group topology on K (being unconditionally closed and of finite index). Therefore, the subgroup mK is contained in every subgroup of K that is *potentially dense* (i.e., dense in some compact Hausdorff group topology on G). A similar comment is valid in the context of Remark 2.8, where both the compact hypothesis and the abelian hypothesis have been discarded.

2.2. The subgroup $\mathbf{fin}(G)$.

In order to better understand the behavior of the subgroup $\mathbf{den}(G)$ of a given topological abelian group G, we "approximate" that subgroup of G from within by means of another subgroup, $\mathbf{fin}(G)$, defined algebraically and without recourse to the topology of G. First, some notation.

Definition 2.10. Let G be an abelian torsion group of bounded order. Then

(a) $exp(G)$, the *finite exponent of* G, is the smallest integer $m > 0$ such that $mG = \{0\}$; and

(b) $eo(G)$, the *essential order of* G, is the smallest integer $m > 0$ such that $|mG| < \omega$.

(c) $\mathbf{fin}(G) := eo(G) \cdot G$.

Note that Definition 2.10 relates only to abelian torsion groups of bounded order. It is convenient, however, to define $exp(G)$, $eo(G)$, and $\mathbf{fin}(G)$ even for abelian groups that are not torsion of bounded order. We set

$$exp(G) = eo(G) = 0 \quad \text{and} \quad \mathbf{fin}(G) := eo(G) \cdot G = 0G = \{0_G\}$$

for such groups.

Remark 2.11. (a) It is obvious that for each abelian torsion group G, these three conditions are equivalent: G is finite; $eo(G) = 1$; $\mathbf{fin}(G) = G$.

(b) For $G = G(p)$ an abelian p-group of bounded order, the cardinal $eo(G)$ is fully determined by the cardinals α_k. For G satisfies exactly one of the three criteria:

(1) $|G| < \omega$;
(2) $\alpha_n \geq \omega$;
(3) $\alpha_n < \omega$ and some $\alpha_k \geq \omega$ $(1 \leq k < n)$,

and in those three cases $eo(G)$ is given, respectively, as follows:

(1) $eo(G) = 1$;
(2) $eo(G) = 0$;
(3) $eo(G) = p^{\overline{k}}$ with $\overline{k} := \max\{k < n : \alpha_k \geq \omega\}$.

(c) To the authors' knowledge, the invariant $eo(G)$ (for G an abelian torsion group of bounded order) was introduced by Berit Nilsen Givens and Kenneth Kunen [33]. They note for such groups G, using the notation of Remark 2.6, that $eo(G)$ is the least common multiple of the integers $\{p^k : \alpha_k \geq \omega\}$.

The following lemma, again in consonance with statements in [33], indicates that each abelian group G has a biggest finite subgroup of the form mG $(m \in \mathbb{Z})$; that subgroup is $\mathbf{fin}(G)$.

Lemma 2.12. *Let G be an abelian group and let $I = I(G) := \{m \in \mathbb{Z} : |mG| < \omega\}$. Then*
(a) *I is an additive subgroup of $\mathbb{Z}$ with generator $eo(G)$; and*
(b) $\mathbf{fin}(G) = \langle \bigcup_{m \in I} mG \rangle$.

Proof. If G is not torsion of bounded order, then $I = \{0\}$ and $\mathbf{fin}(G) = \{0\}$ and the theorem is obvious. Now, assume that G is torsion of bounded order.

(a) For $m_0, m_1 \in \mathbb{Z}$, we have $(m_0 + m_1)G \subseteq \langle m_0G \cup m_1G \rangle$, so if $m_0, m_1 \in I$, then (since a finite subset of G generates a finite subgroup of G) we have $|(m_0 + m_1)G| < \omega$, and hence $m_0 + m_1 \in I$. Clearly, $m \in I \Rightarrow -m \in I$.

(b) According to Definition 2.10(b), $eo(G)$ is the generator (positive, when $|I| > 1$) of the cyclic group I, so

$$m \in I \Rightarrow mG \subseteq eo(G) \cdot G = \mathbf{fin}(G)$$

and (b) follows. □

In parts of Lemma 2.12 we have essentially duplicated the argument of Dikran Dikranjan and Shakhmatov [29, Lemma 4.4]. As those authors remark, it follows from Lemma 2.12 that every abelian group of bounded order satisfies $eo(G)|exp(G)$.

Lemma 2.13. *Let G be an abelian group, $|G| > 1$. Then*

(a) *the following conditions are equivalent:*
  (a$_1$) $|\mathbf{fin}(G)| > 1$*;*
  (a$_2$) *there exists $m \in \mathbb{Z}$ such that $1 < |mG| < \omega$;*
  (a$_3$) *G is bounded torsion and $min\ell_{UK}(G) < \omega$ (i.e., $\ell^p_{UK}(G) < \omega$ for some p).*

(b) $\mathbf{fin}(G/\mathbf{fin}(G)) = \{0\}$*;*

(c) *if H is a group and $h : G \twoheadrightarrow H$ a surjective homomorphism, then $h[\mathbf{fin}(G)] \subseteq \mathbf{fin}(H)$.*

(d) *for every group topology on G, the subgroup $\mathbf{fin}(G)$ is contained in every dense subgroup of G.*

Proof. If $|G| < \omega$, then all statements are obvious (with $eo(G) = 1$ and $G = \mathbf{fin}(G)$), so we assume $|G| \geq \omega$.

(a) That (a$_1$) $\Leftrightarrow$ (a$_2$) is immediate from the definition of $\mathbf{fin}(G)$, and they imply that G is bounded torsion.

(a$_3$) $\Rightarrow$ (a$_2$) Suppose there is $p \in R(G)$ such that $\alpha_{n_p} < \omega$. Let $k := \Pi\{q^{n_q} : p \neq q \in R(G)\}$. Then $m := p^{n_p-1}k$ satisfies $mG = (\mathbb{Z}(p))^{\alpha_{n_p}}$; hence, $1 < |mG| < \omega$.

(a$_2$) $\Rightarrow$ (a$_3$) Let $m \in \mathbb{Z}$ satisfy $1 < |mG| < \omega$. Then there exists $q \in R(G)$ such that $|mG(q)| > 1$, and from the representation given in (2) of Remark 2.6(b), it follows that the relation $q^{n_q}|m$ fails. Then since $|mG(q)| < \omega$, we have $\ell^q_{UK}(G) < \omega$.

(b) Let $H := G/\mathbf{fin}(G)$ and $G \twoheadrightarrow H$ be the quotient homomorphism. According to Lemma 2.12(b), it suffices to show that each $m \in \mathbb{Z}$ such that $|mH| < \omega$ satisfies $mH = \{0\}$. Given such m, we have $|mH| = |h[mG]| < \omega$, and hence $|mG| < \omega$ (since $|\ker(h)| = |\mathbf{fin}(G)| < \omega$), so

$$mG \subseteq \mathbf{fin}(G) \text{ and } mH = h[mG] = \{0\}.$$

(c) According to Lemma 2.12(b), it suffices to show $h[mG] \subseteq \mathbf{fin}(H)$ for every $m \in \mathbb{Z}$ such that $|mG| < \omega$. That is obvious, since $h[mG]$ is finite for such m, hence is contained in $\mathbf{fin}(H)$.

(d) is immediate from Theorem 2.7. □

3. Statement of the Main Results

Here with minimal commentary we state our principal results. Proofs are given in sections 5 (the case where K is torsion) and 6 (the case where $r(K) > 0$).

Lemma 2.13(d) asserts for each (abelian) topological group that $\mathbf{fin}(G) \subseteq \mathbf{den}(G)$, with $\mathbf{den}(G)$ defined as above: $\mathbf{den}(G) := \bigcap \mathcal{D}(G)$. We show below (Theorem 6.10(b)) that this inclusion becomes an equality for every compact abelian group G. More specifically, we prove the following two results.

Theorem A. *Every compact abelian group K admits a pair of almost disjoint dense subgroups; hence $|\mathbf{den}(K)| < \omega$ for such K.*

Obviously, every fragmentable group K satisfies $\mathbf{den}(K) = \{e\}$. Our next principal result shows that for a compact abelian group K the converse holds.

Theorem B. *Let K be a compact abelian group. Then*

(a) *$K \in \mathcal{F}(2)$ if and only if $\mathbf{den}(K) = \{0\}$;*

(b) *there exists $m \in \mathbb{Z}$ such that $1 < |mK| < \omega$ and $\mathbf{den}(K)$ is the biggest among all finite subgroups of K of the form mK, i.e., $\mathbf{den}(K) = \mathbf{fin}(K)$; and*

(c) *there exist $D_1, D_2 \in \mathcal{D}(K)$ such that $\mathbf{den}(K) = D_1 \cap D_2$.*

Remark 3.1. (a) In view of Definition 2.10, Theorem B(c) shows that the subgroup $\mathbf{den}(K)$ of a compact abelian group K has a purely algebraic description, i.e., that subgroup is independent of the compact group topology taken on K. Two comments will serve to show that this remark is not vacuous. First, in a model of ZFC with $2^\omega = 2^{(\omega^+)}$ the abelian group $G = \{0,1\}^\omega$ admits compact group topologies $\mathcal{T}_0$ and $\mathcal{T}_1$ which are very different; for example, one may have $w(G, \mathcal{T}_0) = \omega$ and $w(G, \mathcal{T}_1) = \omega^+$. Secondly, even in models of ZFC which are "well behaved," for example satisfying GCH, many infinite compact abelian groups $(K, \mathcal{T})$ admit discontinuous automorphisms and from these it is easy to define compact group topologies on K which differ from $\mathcal{T}$.

(b) A restatement of Theorem B(c) is that the density nucleus $\mathbf{den}(K)$ of an arbitrary compact abelian group K coincides with the intersection of just *two* (appropriately chosen) dense subgroups.

Now we show that for a compact abelian group K the fragmentation number $\mathfrak{f}(K)$ (as in Definition 2.3) exists and is well defined. Further, we compute its value explicitly and in concrete form, in terms of algebraic invariants of K. To do that, we consider separately the torsion case and the case $r(K) > 0$.

Theorem C. *Let $K \in \mathcal{F}(2)$ be a compact abelian torsion group. Then*

(a) *$\mathfrak{f}(K) = \ell_{UK}(K)$, if K is a p-group;*

(b) *$\mathfrak{f}(K) = min\ell_{UK}(K)$ $(= \min\{\ell^p_{UK}(K) : p \in R(K)\})$ in the general case.*

Note that $K \in \mathcal{F}(2)$ is a necessary hypothesis in Theorem C: If $K \notin \mathcal{F}(2)$, then $\mathfrak{f}(K) = 1$, while $\ell_{UK}(K)$, though finite, may be arbitrarily large (so $\mathfrak{f}(K) = \ell_{UK}(K)$ can fail in item (a)).

Item (b) can be easily deduced from item (a) (we do this in the beginning of the next section), whereas (a) is much less trivial and will be proved only in §4.

Corollary C. *Let K be a compact abelian torsion group. Then*
(a) $\mathfrak{f}(K) = 1$ *if and only if* $K \notin \mathcal{F}(2)$*; and*
(b) *if* $\mathfrak{f}(K) > 1$*, then* $\mathfrak{f}(K)$ *is infinite, specifically* $\mathfrak{f}(K) = min\ell_{UK}(K)$.

It follows from the above theorem that, since infinite Ulm-Kaplansky invariants have the form 2^{ϵ_k}, if K in Corollary C has $\mathfrak{f}(K) > 1$, then $\mathfrak{f}(K)$ is an infinite cardinal of exponential type (so $\mathfrak{f}(K) \geq \mathfrak{c}$, in particular).

Theorem D. *If K is a compact abelian group such that $r(K) > 0$, then $\mathfrak{f}(K) = r(K)$. Moreover, the following conditions are equivalent:*
(a) *K is $r(K)$-free-fragmentable, i.e., $K \in \mathcal{F}_f(r(K))$;*
(b) $K \in \mathcal{F}_f(1)$*;*
(c) $d(K) \leq r(K)$*;*
(d) $\mathfrak{f}(K) = \mathfrak{f}_f(K)$.

The inequality $\mathfrak{f}(K) \leq r(K)$ will be proved in Lemma 6.3. The more difficult inequality $\mathfrak{f}(K) \geq r(K)$ will be deduced in Corollary 6.6 in the case of w-divisible groups. The general case requires additional argument, given in Theorem 6.11.

Corollary D1. *A compact abelian group K is maximally fragmentable if and only if either*
(a) *K is torsion and $min\ell_{UK}(K) = |K|$, or*
(b) $r(K) = |K|$.

This corollary has the following immediate consequence: *Any maximally fragmentable compact abelian group is also maximally ad-fragmentable.* Indeed, if K is a maximally fragmentable compact abelian group, then any family of dense subgroups witnessing $K \in \mathcal{F}(|K|)$ witnesses also $K \in \mathcal{F}_{ad}(|K|) = \mathcal{F}_{ad}(|K|^\omega)$, since $|K| = |K|^\omega$.

It turns out that a compact abelian group K is κ-fragmentable if and only if it is κ-pairwise-fragmentable.

Corollary D2. *Let $\kappa > 0$ be a cardinal and let K be a compact abelian group. Then*

$$K \in \mathcal{F}_2(\kappa) \Longleftrightarrow K \in \mathcal{F}(\kappa). \tag{4}$$

Since the invariants $\mathfrak{f}(K)$ and $\mathfrak{f}_2(K)$ are well defined for compact abelian groups K, (4) has this consequence:

$$\mathfrak{f}(K) = \mathfrak{f}_2(K). \tag{5}$$

The next example shows that equality (5) can fail for a precompact abelian group K for which $\mathfrak{f}(K)$ and $\mathfrak{f}_2(K)$ are well defined.

Example 3.2. Let A be a free set in $\mathbb{T}$ and let $G_A := \langle A \rangle$. Then
(a) G_A is free with $r(G_A) = |A|$, so for every κ, one has

$$G_A \in \mathcal{F}_f(\kappa) \Leftrightarrow G_A \in \mathcal{F}(\kappa) \text{ and } G_A \in \mathcal{F}_2(\kappa) \Leftrightarrow G_A \in \mathcal{F}_{ad}(\kappa).$$

So the number $\mathcal{F}_2(\kappa)$ exists if and only if the number $\mathcal{F}_{ad}(\kappa)$ exists; in such a case they are equal and $\leq |G_A|$.
(b) each nonzero subgroup of G_A is dense in $\mathbb{T}$, so in G_A as well; so for every κ, one has

$$G_A \in \mathcal{F}_f(\kappa) \Leftrightarrow G_A \in \mathcal{F}(\kappa) \Leftrightarrow \kappa \leq |A| = r(G_A). \tag{$*$}$$

(c) if $|A| > 1$, then for each two-element subset $\{a, b\}$ of A and for $0 < n_0 < n_1 < \omega$, one checks easily that the subgroups $H_i := \langle a + n_i b \rangle$ $(i = 0, 1)$ satisfy $H_0 \cap H_1 = \{0\}$.

Consequently, one can compute the fragmentation numbers as follows.

- If $|A| = 1$, then $\mathfrak{f}(G_A) = \mathfrak{f}_f(G_A) = \mathfrak{f}_2(G_A) = \mathfrak{f}_{ad}(G_A) = 1$.
- If $1 < |A| < \omega$, one has $G_A \in \mathcal{F}_2(\omega)$ from (c), while $\mathfrak{f}(G_A) = \mathfrak{f}_f(G_A) = |A|$ from (b). So from (a) and (*), one can deduce that

  $$\mathfrak{f}(G_A) = \mathfrak{f}_f(G_A) = |A| < \omega = \mathfrak{f}_2(G_A) = \mathfrak{f}_{ad}(G_A).$$

- If A is infinite, one has

  $$\mathfrak{f}(G_A) = \mathfrak{f}_f(G_A) = \mathfrak{f}_{ad}(G_A) = \mathfrak{f}_2(G_A) = |A|,$$

  using (a) and the fact that $\mathcal{F}_f(\kappa) \subseteq \mathcal{F}(\kappa) \subseteq \mathcal{F}_2(\kappa) \subseteq \mathcal{F}_{ad}(\kappa)$ holds for all κ.

4. Properties of the Fragmentation Numbers and the Classes $\mathcal{F}_f(\kappa)$, $\mathcal{F}(\kappa)$, $\mathcal{F}_2(\kappa)$, $\mathcal{F}_{ad}(\kappa)$

We begin with a simple lemma.

Lemma 4.1. *Let $\kappa > 0$ be a cardinal, let $\{G_i : i \in I\}$ be a set of topological groups, and let $G := \prod_{i \in I} G_i$. Then*
(a) *if each $G_i \in \mathcal{F}_f(\kappa)$, then $G \in \mathcal{F}_f(\kappa)$;*
(b) *if each $G_i \in \mathcal{F}(\kappa)$, then $G \in \mathcal{F}(\kappa)$;*
(c) *if each $G_i \in \mathcal{F}_2(\kappa)$, then $G \in \mathcal{F}_2(\kappa)$;*
(d) *if $|I| < \omega$ and each $G_i \in \mathcal{F}_{ad}(\kappa)$, then $G \in \mathcal{F}_{ad}(\kappa)$.*

Proof. Let $\{G_i(\eta) : \eta < \kappa\}$ be a set of κ-many dense subgroups of G_i as hypothesized (independent and free in (a), independent in (b), pairwise independent in (c), almost disjoint in (d)), and for $\eta < \kappa$, set $G(\eta) := \bigoplus_{i\in I} G_i(\eta)$. Then $\{G(\eta) : \eta < \kappa\}$ is a family as required in G. □

Because of its utility, we note explicitly this corollary to Lemma 4.1.

Corollary 4.2. *Let $\kappa > 1$ and let K be a compact abelian torsion group. Then*

(a) $K \in \mathcal{F}(\kappa) \;\Leftrightarrow\;$ *each* $K(p) \in \mathcal{F}(\kappa)$ $(p \in R(K))$;
(b) $K \in \mathcal{F}_2(\kappa) \;\Leftrightarrow\;$ *each* $K(p) \in \mathcal{F}_2(\kappa)$ $(p \in R(K))$; *and*
(c) $K \in \mathcal{F}_{ad}(\kappa) \;\Leftrightarrow\;$ *each* $K(p) \in \mathcal{F}_{ad}(\kappa)$ $(p \in R(K))$.

Proof. We prove (a), the other proofs being very similar.

($\Leftarrow$) Follows from Lemma 4.1(b).

($\Rightarrow$) Let $\{D_i : i \in I\}$ be an independent family of free dense subgroups of K with $|I| = \kappa$. Note that for $p \in R(K)$ one has $D_i(p) = D_i \cap K(p)$. Since $D_i = \bigoplus_{p\in R(K)} D_i(p)$, this proves that $D_i(p)$ is a dense subgroup of $K(p)$, being the projection of the dense subgroup D_i of K. □

Remark 4.3. (a) Corollary 4.2 gives Theorem C(b) from Theorem C(a), so in the sequel concerning Theorem C we need to prove only (a).

(b) Clearly the classes cited in Lemma 4.1 are also closed under passage to weaker group topologies, and under passage to dense overgroups.

Lemma 4.4. *Let $\kappa > 1$ and G be a topological abelian group such that $G \in \mathcal{F}_2(\kappa)$. Then $\kappa \le |mG|$ for each $m < \omega$ such that $mG \ne \{0\}$.*

Proof. Let $\{D_\eta : \eta < \kappa\}$ be a pairwise independent family of dense subgroups of G. From $mG \ne \{0\}$ follows $mD_\eta \ne \{0\}$, and for distinct $\eta, \xi < \kappa$, we have $mD_\eta \cap mD_\xi \subseteq D_\eta \cap D_\xi = \{0\}$. Thus, for every $\eta < \kappa$, there is $x_\eta \in mD_\eta \setminus \bigcup_{\eta\ne\xi<\kappa} mD_\xi$, and we have $|mG| \ge |\{x_\eta : \eta < \kappa\}| = \kappa$, as required. □

Corollary 4.5. *Let $\kappa > 1$ and let K be an infinite compact abelian p-group, say $exp(K) = p^n$ $(p \in \mathbb{P},\ n > 0)$. If $K \in \mathcal{F}_2(\kappa)$, then $\ell_{UK}(K) = |p^{n-1}K|$ is infinite and $\kappa \le |p^{n-1}K|$.*

Proof. If $\ell_{UK}(K)$ is finite, then $\mathbf{fin}(K) \ne \{0\}$, so $\mathbf{den}(K) \ne \{0\}$ as well. This contradicts $K \in \mathcal{F}_2(\kappa)$. This proves that $\ell_{UK}(K) = |p^{n-1}K|$ is infinite.

Since $p^{n-1}K = ((\mathbb{Z}(p))^{\epsilon_n} \ne \{0\}$, the second assertion follows from Lemma 4.4. □

The implication $K \in \mathcal{F}_2(\kappa) \Rightarrow \kappa \le |p^{n-1}K|$ of Corollary 4.5 suggests a parallel statement for the class $\mathcal{F}_{ad}(\kappa)$. We have been unable to settle this issue, so we leave it as an open question.

Question 4.6. Let K be an infinite compact abelian p-group, say $exp(K) = p^n$ ($p \in \mathbb{P}$, $n > 0$). Does $K \in \mathcal{F}_{ad}(\kappa)$ for some $\kappa > 1$ imply $\kappa \leq |p^{n-1}K|$?

Note that a positive answer to Question 4.6 will imply, along with Theorem 5.1, that a maximally ad-fragmentable infinite compact abelian p-group is maximally fragmentable. In other words, the properties "maximally fragmentable" and "maximally ad-fragmentable" coincide for infinite compact abelian p-groups.

Theorem 4.7. *Let G be a topological group and let $h : G \twoheadrightarrow N$ be a quotient homomorphism. Then*

(a) *if L is a dense subgroup of N, then $h^{-1}(L)$ is a dense subgroup of G;*
(b) *$h^{-1}[\mathbf{den}(G)] \subseteq \mathbf{den}(N)$; and*
(c) *$h[\mathbf{den}(G)] = \mathbf{den}(N)$.*

Proof. h is an open function, so (a) is immediate. (b) then follows. The relation $\subseteq$ of (c) follows from (b), while $\supseteq$ is obvious (since h is surjective). □

5. Computing the Numbers $\mathfrak{f}(K)$: The Torsion Case

First we prove item (a) of Theorem C in the case when the leading Ulm-Kaplansky invariant $\ell_{UK}(K)$ of the compact abelian p-group K is *dominating* in the sense that $\ell_{UK}(K) = |K|$. According to Corollary 4.5, maximally fragmentable compact abelian p-groups have this property. Now we show that the equality $\ell_{UK}(K) = |K|$ is also sufficient for a compact abelian p-group to be maximally fragmentable.

Theorem 5.1. *Let $p \in \mathbb{P}$ and let K be an infinite abelian compact p-group with $exp(K) = p^n$. Then the following are equivalent:*

(a) *$|p^{n-1}U| = |K|$ for every nonempty open subset U of K;*
(b) *K admits an independent family of $|K|$-many dense subgroups, each isomorphic to $\bigoplus_{|K|} \mathbb{Z}(p^n)$;*
(c) *K is maximally fragmentable, i.e., $K \in \mathcal{F}(|K|)$;*
(d) *$2^{\epsilon_n} = |K|$; and*
(e) *$K = B \times (\mathbb{Z}(p^n))^{\epsilon_n}$ with $2^{\epsilon_n} = |K|$ and with B a compact p-group such that $p^{n-1}B = \{0\}$.*

Proof. (b) $\Rightarrow$ (c) is trivial and (c) $\Rightarrow$ (d) follows from Corollary 4.5.

(d) $\Rightarrow$ (e) is clear with $B := \prod_{1 \leq k < n} (\mathbb{Z}(p^k))^{\epsilon_k}$.

(e) $\Rightarrow$ (a) Finitely many translates of U cover K, so finitely many translates of $p^{n-1}U$ cover $p^{n-1}K \cong (\mathbb{Z}(p))^{\epsilon_n}$, so

$$|p^{n-1}U| = |(\mathbb{Z}(p))^{\epsilon_n}| = 2^{\epsilon_n} = |K|.$$

(a) $\Rightarrow$ (b) We first establish this statement:

(*) *If X is a subgroup of K with $|X| < |K|$ and if U is a nonempty open subset of K, then there exists $s \in U$ such that $\langle s\rangle \cap X = 0$ and $o(s) = p^n$.*

From (a) we have $|p^{n-1}U| = |K|$, so $|X| < |K|$ yields $|p^{n-1}U \setminus X| = |K|$. Thus, there are $|K|$-many $s \in U$ such that $p^{n-1}s \notin X$, each with $o(s) = p^n$. Since $o(p^{n-1}s) = p$, the cyclic subgroup $C := \langle p^{n-1}s\rangle$ has no proper subgroups; hence, $C \cap X = \{0\}$. Then since C is essential in $\langle s\rangle$, we have $\langle s\rangle \cap X = \{0\}$ and (*) is proved.

One checks easily, perhaps using Theorem 1.2, that the family $\mathcal{T}^{\#}$ of nonempty open subsets of K satisfies $|\mathcal{T}^{\#}| = |K|$. Finitely many translates of each $U \in \mathcal{T}^{\#}$ cover K, so $|U| = |K|$ for each such U.

Now we introduce two indexings of $\mathcal{T}^{\#}$: $\{U_i : i \in I\}$ is a faithful indexing of $\mathcal{T}^{\#}$ (with $|I| = |K|$), and $\{U_\eta : \eta < |K|\}$ is an indexing of $\mathcal{T}^{\#}$ in which each element of $\mathcal{T}^{\#}$ appears $|K|$-many times. By recursion we will define for $\eta < |K|$ subgroups X_η of K and points $s_\eta \in K$ such that

(a$_\eta$) $o(s_\eta) = p^n$;
(b$_\eta$) $\langle s_\eta\rangle \cap X_\eta = \{0\}$; and
(c$_\eta$) $X_\eta := \langle\{s_\xi : \xi < \eta\}\rangle$

hold true for every $\eta < |K|$.

Let $X_0 = \{0\}$, and using (*), choose $s_0 \in U_0$ such that $o(s_0) = p^n$.

Suppose now that $\eta < |K|$ and that s_ξ and X_ξ have been defined for all $\xi < \eta$ satisfying (a$_\xi$), (b$_\xi$), and (c$_\xi$) for all $\xi < \eta$. Set $X_\eta := \langle\{s_\xi : \xi < \eta\}\rangle$. Then $|X_\eta| < |K|$, so by (*) there is $s_\eta \in U_\eta$ such that $\langle s_\eta\rangle \cap X_\eta = \{0\}$ and $o(s_\eta) = p^n$. Then X_η and s_η are defined for all $\eta < |K|$, satisfying (a$_\eta$), (b$_\eta$), and(c$_\eta$).

We set $S := \{s_\eta : \eta < |K|\}$ and $Y := \bigcup_{\eta<|K|} X_\eta = \langle S\rangle$. Then S is a free set, $\langle s_\eta\rangle \cong \mathbb{Z}(p^n)$ for each $s_\eta \in S$, $|S \cap U_i| = |K|$ for each $U_i \in \mathcal{T}^{\#}$, and

$$Y = \bigoplus_{s_\eta<|K|} \langle s_\eta\rangle \cong \bigoplus_{|K|} \mathbb{Z}(p^n).$$

Recall now the so-called Disjoint Refinement Lemma:

(**) *If $\{A_i : i \in I\}$ is a (not necessarily faithfully indexed) set with $|I| = \lambda \geq \omega$ and with each $|A_i| \geq \lambda$, then there are pairwise disjoint sets B_i ($i \in I$) such that $|B_i| = \lambda$ and $B_i \subseteq A_i$ for each $i \in I$.*

(This combinatorial result has been noted independently by several authors, perhaps for the first time by Casimir Kuratowski [40]; for additional references and a proof, see also [19, Lemma 7.5].)

Recalling the faithful indexing $\{U_i : i \in I\}$ of $\mathcal{T}^{\#}$ with $|I| = |K|$, for $i \in I$ set $A_i := S \cap U_i$. Then $|A_i| = |K|$ for each $i < |K|$ and there is a family $\{B_i : i \in I\}$ as given by (**) (with $\lambda := |I|$). Then with

$$D_i := \langle B_i \rangle = \bigoplus_{s \in B_i} \langle s \rangle \cong \bigoplus_{s \in B_i} \mathbb{Z}(p^n) \cong \bigoplus_{|K|} \mathbb{Z}(p^n) \quad \text{for} \quad i \in I,$$

the family $\{D_i : i < |K|\}$ is as required. □

As far as the classes $\mathcal{F}_{ad}(-)$ are concerned, the next combinatorial statement becomes relevant.

Theorem 5.2. *Let X be a set such that $|X| = \kappa \geq \omega$. Then*
(a) *there is an almost disjoint family $\mathcal{A} \subseteq \mathcal{P}(X)$ such that $|\mathcal{A}| = \kappa^\omega$, and*
(b) *there is no almost disjoint family $\mathcal{B} \subseteq \mathcal{P}(X)$ such that $|\mathcal{B}| > \kappa^\omega$.*

For $\kappa = \omega$ this was proved by W. Sierpiński [47]; the full statement and substantial additional generalizations (not needed here) are given by Alfred Tarski [48](Théorème 7). See also [19, Theorem 12.2; Notes for §12] for proofs and extensive relevant bibliographic citations.

Corollary 5.3. *Let $p \in \mathbb{P}$ and let K be a compact abelian p-group with* $\exp(K) = p^n$. *If K satisfies one (hence all) of conditions (a)–(e) of Theorem 5.1, then $\mathfrak{f}_f(K) = 0$ and $\mathfrak{f}(K) = \mathfrak{f}_2(K) = \mathfrak{f}_{ad}(K) = |K|$.*

Proof. That $\mathfrak{f}_f(K) = 0$ for every torsion group K is obvious. Clearly for every group K we have $\kappa \leq |K|$ if $K \in \mathcal{F}_2(\kappa)$, so Theorem 5.1 shows $\mathfrak{f}(K) = \mathfrak{f}_2(K)$. To see that also $\mathfrak{f}_{ad}(K)$ is well-defined in the present case, indeed with $\mathfrak{f}_{ad}(K) = |K|$, we make recourse to Theorem 5.2 as follows. In the present case we have $K \in \mathcal{F}_{ad}(|K|)$ from Theorem 5.1, while from (*) it follows that $K \in \mathcal{F}(\lambda)$ fails for $\lambda > |K|^\omega$. Since $|K|^\omega = (2^{w(K)})^\omega = 2^{w(K)} = |K|$ (from Theorem 1.2), we have $\mathfrak{f}_{ad}(K) = |K|$, as asserted. □

Remark 5.4. (a) From (d) in Theorem 5.1 or otherwise, we have $|K| = 2^{\epsilon_n}$ and, as indicated earlier, we also have $|K| = 2^{w(K)}$. Those relations do not in ZFC imply that $\epsilon_n = w(K)$, although in the present case we have $\epsilon_n = w(p^{n-1}K) \leq w(K)$.

(b) It is not difficult to see that if a topological group $K = (K, \mathcal{T})$ is as in Theorem 5.1 and $p^{n-1}K$ is non-metrizable, then the five conditions (a)–(e), suitably modified and interpreted, remain equivalent when $\mathcal{T}$ is replaced by the topology $\mathcal{T}_\delta$ determined by the G_δ-subsets of $(K, \mathcal{T})$ in items (a)–(c). (In the terminology favored by some authors, the space $(K, \mathcal{T}_\delta)$ is the *P-space modification of* $(K, \mathcal{T})$. Of course, K itself is no longer compact in the topology $\mathcal{T}_\delta$.) To be specific, conditions (a), (b), and (c) remain equivalent as stated *verbatim* when interpreted to pertain to the $\mathcal{T}_\delta$ topology. In this connection, two brief comments are in order:

(1) $|\mathcal{B}| \leq |\mathcal{T}^\delta|^\omega = |K|^\omega = |K|$, since $|K| = 2^{w(K)}$.

(2) Each nonempty set U open in $((\mathbb{Z}(p))^{\epsilon_n}, \mathcal{T}_\delta)$ satisfies $|U| = 2^{w(K)} = |K|$. (This equality holds true in every non-metrizable not necessarily abelian compact group K.)

Since these considerations play no role in this paper, and we plan to explore these and related issues in detail in a later communication [7], we omit the details of their proofs.

(c) The upshot of the discussion in (b) is the following statement (see also [7]), whose generalization to arbitrary compact abelian torsion groups will by now be obvious to the reader; one uses here the fact, a frequently noted consequence of [22, Theorem 1.2], that a dense subgroup D of a compact group K is pseudocompact if and only if D is G_δ-dense in K.

Let $p \in \mathbb{P}$ and let K be a compact abelian p-group for which $exp(K) = p^n$ such that $\epsilon_n = w(p^{n-1}K) > \omega$ and $2^{\epsilon_n} = |K|$. Then K admits an independent family of $|K|$-many dense pseudocompact subgroups, each isomorphic to $\bigoplus_{|K|} \mathbb{Z}(p^n)$.

Now we give two corollaries to Theorem 5.1. The first proves Theorem A for torsion groups, the second completes the proof of Theorem C.

Theorem 5.5. *Let K be an infinite compact abelian torsion group. Then $K \in \mathcal{F}_{ad}(2)$.*

Proof. By Corollary 4.1(d), it suffices to show $K(p) \in \mathcal{F}_{ad}(2)$ for each $p \in R(K)$. We assume then that K is a p-group, i.e., $K = K(p)$.

As in relation (2) of Remark 2.6(b), we write $K = \prod_{k=1}^n K_k$, where $K_k = (\mathbb{Z}(p^k))^{\epsilon_k}$. With $A := \{k : 1 \leq k \leq n, |K_k| \geq \omega\}$, we have $A \neq \emptyset$, and (from Theorem 5.1), we have $K_k \in \mathcal{F}(|K_k|) \subseteq \mathcal{F}_{ad}(2)$ for each $k \in A$. Since $\Pi_{k \notin A} K_k$ is finite and $\Pi_{k \in A} K_k \in \mathcal{F}_{ad}(2)$ by Lemma 4.1(d), we have

$$K = \prod_{k=1}^{n} K_k = \left(\prod_{k \in A} K_k\right) \times \left(\prod_{k \notin A} K_k\right) \in \mathcal{F}_{ad}(2). \qquad \square$$

Proof of Theorem C. As noted in Remark 4.3(a), it suffices to prove that $\mathfrak{f}(K) = \ell_{UK}(K)$ when $K = K(p)$ is a p-group.

We have $\mathfrak{f}(K) \leq |p^{n-1}K|$ by Corollary 4.5. To prove $\mathfrak{f}(K) \geq |p^{n-1}K|$, we argue by induction on n. The case $n = 1$ follows from Theorem 5.1, since $\ell_{UK}(K) = |K|$ in that case. Assume that $n > 1$. In case $\ell_{UK}(K) = |K|$, Theorem 5.1 applies again. Assume that $\ell_{UK}(K) < |K|$. Then we can write $K = K_1 \times K_2$, where $p^{n-1}K_1 = \{0\}$, $\beta := \ell_{UK}(K) = \ell_{UK}(K_2) = |K_2| < \ell_{UK}(K_1)$. Then Theorem 5.1 applied to K_2 gives

$\mathfrak{f}(K_2) \geq \ell_{UK}(K_2) = \beta$, and the same theorem, together with the inductive hypothesis applied to the group K_1 with $exp(K_1) \leq p^{n-1}$, gives $\mathfrak{f}(K_1) \geq \ell_{UK}(K_1) > \beta$. Hence, $\min\{\mathfrak{f}(K_1), \mathfrak{f}(K_2)\} = \mathfrak{f}(K_2) = \beta = \ell_{UK}(K)$ and, from Lemma 4.1(b), we have $\mathfrak{f}(K) \geq \min\{\mathfrak{f}(K_1), \mathfrak{f}(K_2)\} = \ell_{UK}(K)$. □

Now we resolve fully the question: Which compact abelian torsion groups are in $\mathcal{F}(2)$?

Theorem 5.6. *Let K be a compact abelian torsion group. Then the following conditions are equivalent:*

(a) $|\mathbf{fin}(K)| > 1$*;*
(b) *there is $m \in \mathbb{N}$ such that $K[m] := \{x \in K : mx = 0\}$ is a proper open subgroup of K;*
(c) *at least one leading Ulm-Kaplansky invariant of K is finite;*
(d) $K \notin \mathcal{F}(2)$.

Proof. (a) restates (a_1) of Lemma 2.13(a); an integer m is as in (b) if and only if $1 < |K/K[m]| < \omega$ so, since $K/K[m] \cong mK$, (b) restates (a_2) of Lemma 2.13(a); and since K is bounded torsion, (c) restates (a_3) of Lemma 2.13(a). Thus, the equivalence of (a), (b) and (c) is exactly the equivalence given in Lemma 2.13(a).

We show (a) ⇒ (d) and (d) ⇒ (c).

(a) ⇒ (d) Lemma 2.13(d) shows $\mathbf{fin}(K) \subseteq \mathbf{den}(K)$, and this implication is immediate.

(d) ⇒ (c) We show that if (c) fails, then $K \in \mathcal{F}(2)$. By Corollary 4.2(b), it suffices to show for each $p \in R(K)$ that $K(p) \in \mathcal{F}(2)$, so we assume that $K = K(p) = \prod_{1 \leq k \leq n} (\mathbb{Z}(p^k))^{\epsilon_k}$, with $\epsilon_n \geq \omega$. With

$$A := \{k < n : \epsilon_k < \omega\} \quad \text{and} \quad K' := \prod_{k \in A \cup \{n\}} (\mathbb{Z}(p^k),$$

we have from Theorem 5.1((e) ⇒ (c)) that

$$K' \in \mathcal{F}(|K'|) \subseteq \mathcal{F}(2) \ \ (\text{with } \ |K'| = 2^{\epsilon_n});$$

similarly, since $\epsilon_k \geq \omega$ for each k such that $1 \leq k \leq n, k \notin A, k \neq n$, we have $(\mathbb{Z}(p^k))^{\epsilon_k} \in \mathcal{F}(2)$ for each such k. Hence,

$$K = K(p) = \prod_{1 \leq k \leq n} (\mathbb{Z}(p^k))^{\epsilon_k} \in \mathcal{F}(2),$$

again by Corollary 4.2(b). □

The following definitions will be useful as we approach the proof of Theorem D.

Definition 5.7. For an abelian torsion group K of bounded order, K_{sing} and K_{inf} are defined as follows.

Case 1: $K = K(p)$, say $K = \prod_{1\leq k\leq n} (\mathbb{Z}(p^k))^{\epsilon_k}$.
If $|K| < \omega$, then $K_{sing} := K$ and $K_{inf} := \{0\}$;
if $\epsilon_n \geq \omega$, then $K_{sing} := \{0\}$ and $K_{inf} := K$; and
if $|K| \geq \omega$ and $\epsilon_n < \omega$ and $\overline{k} := \max\{k < n : \epsilon_k \geq \omega\}$, then $K_{sing} := \prod_{\overline{k}<k\leq n} (\mathbb{Z}(p^k))^{\epsilon_k}$, and $K_{inf} := \prod_{k\leq \overline{k}} (\mathbb{Z}(p^k))^{\epsilon_k}$.

Case 2 (the general case): $K = \prod_{p\in R(K)} K(p)$. Then

$$K_{sing} := \prod_{p\in R(K)} (K(p))_{sing}, \quad \text{and} \quad K_{inf} := \prod_{p\in R(K)} (K(p))_{inf}.$$

We record the salient properties of K_{sing} and K_{inf} for compact abelian torsion groups K.

Theorem 5.8. *Let K be a compact abelian torsion group. Then*
(a) *K_{sing} and $K_{\rm inf}$ are complementary direct summands of K;*
(b) *K_{sing} is finite and contains* **fin**(K) *as an essential subgroup; and*
(c) *K_{inf} is open in K.*

Proof. Clearly both (a) and (b) hold for $K = K(p)$, so they hold in general.

From the definition, the group K_{inf} is closed in K, and from (a) and (b), it has finite index in K; (c) then follows. □

Example 5.9. As indicated in Theorem 5.8, for K as hypothesized in Definition 5.7 the subgroup K_{sing} is a finite direct summand of K containing **fin**(K) as an essential subgroup. We show by a simple example that there may be other such subgroups.

Note first that the group $G := \mathbb{Z}(4) \times \mathbb{Z}(2)$ splits naturally in two different ways:
$G = \langle\{(1,0)\}\rangle \times \langle\{(0,1)\}\rangle$ and $G = \langle\{(1,1)\}\rangle \times \langle\{(0,1)\}\rangle$, with $\langle\{(1,1)\}\rangle \cong \mathbb{Z}(4)$ and $\langle\{(0,1)\rangle \cong \mathbb{Z}(2)$.

Then, with $L := (\mathbb{Z}(2))^\omega$ and $K := G \times L$, it is clear that (at least) two finite direct summands of K contain $\mathbf{fin}(K) = 2K = \langle\{(2,0,0_L)\}\rangle$ as an essential subgroup:

$$K_{sing} = \mathbb{Z}(4) \times \{0\} \times \{0_L\} = \langle\{(1,0,0_L)\}\rangle, \text{and}\, S := \langle\{(1,1,0_L)\}\rangle.$$

6. Computing the Numbers $\mathfrak{f}(K)$: The Case $r(K) > 0$

Definition 6.1. Let G be a topological abelian group.

(a) G is *w-divisible* if $w(mG) = w(G) \geq \omega$ whenever $1 < m \in \mathbb{N}$;
(b) G is *div-metrizable* if mG is metrizable for some positive $m \in \mathbb{N}$;

(c) The *divisible weight* (*d-weight*) of G is $w_d(G) := \min\{w(mG) : 1 < m \in \mathbb{N}\}$.

This terminology and definition differ from those given originally in [25]. There the div-metrizable groups were called *singular*, and, for technical reasons, the present condition (a) was given as $w(mG) = w(mG) > \omega$ (thus excluding metrizable groups from the roster of w-divisible groups). That restriction from [25] is not helpful here, hence the present formulation.

Clearly a discrete abelian group G is w-divisible if and only if $|mG| = |G| \geq \omega$ whenever $0 < m \in \mathbb{N}$.

Lemma 6.2. *Let K be a compact abelian group such that $r(G) > 0$. Then*

(a) $r(K) = 2^{w_d(K)}$*; thus* $r(K) = |K| = 2^{w(K)} > w(K)$ *if K is w-divisible.*
(b) *there exists a positive $m_0 \in \mathbb{N}$ such that $K_0 := m_0K$ is w-divisible; thus, $r(K) = r(K_0) = 2^{w(m_0K)}$.*
(c) $\min\{|mK| : 0 < m \in \mathbb{N}\} = 2^{w(m_0K)} = r(K)$.

Proof. (a) If K is not div-metrizable, this equality is Corollary 3.9 from [25]. If K is div-metrizable, then mG is metrizable for some positive $m \in \mathbb{N}$. This implies $w_d(K) = \omega$, since nK is infinite when $0 < n \in \mathbb{N}$. On the other hand, $r(K) = r(mK) = \mathfrak{c} = 2^{w_d(K)}$, as mK is an infinite compact metrizable group and the quotient K/mK is torsion.

(b) $\min\{w(mK) : m \in \mathbb{N}\} = w_d(K)$ is attained, say with $m = m_0$. Then $K_0 := m_0K$ is w-divisible, so $|K_0| = r(K_0) = r(K) = 2^{w(m_0K)}$ by (a).

(c) then follows, since mK is infinite and $|mK| = 2^{w(mK)}$ for all $m \in \mathbb{N}$. □

With reference to item (c) of Lemma 6.2, we note that the equality $\min\{|mK| : m \in \mathbb{N}\} = r(K)$ can fail if the group is not compact. Indeed, every divisible torsion group K satisfies $\min\{|mK| : m \in \mathbb{N}\} = |K| \neq 0 = r(K)$.

Lemma 6.3. *Let K be a compact abelian group such that $r(K) > 0$. Then $\mathfrak{f}(K) \leq r(K)$.*

Proof. We have from Theorem 1.4 that $r(K) \geq \mathfrak{c}$. We have also $mK \neq 0$ when $0 < m \in \mathbb{N}$, so from Lemma 4.4 and Lemma 6.2, we have

$$\mathfrak{f}(K) \leq \min\{|mK| \ : \ m \in \mathbb{N}\} = r(K).$$ □

We make substantial use also of the following major result on w-divisible groups from [25].

Theorem 6.4 ([25]). *Every compact abelian group K splits as a direct product $K = K_{tor} \times K_w$, where K_{tor} is a compact torsion group and the compact group K_w is w-divisible.*

Our next theorem compares as follows with Theorem 1.5 cited above (p. 330) from [14, Theorem 2.8]. That result requires $\omega < \kappa = r(G)$ and gives only the conclusion $G \in \mathcal{F}(\kappa)$; here, assuming $r(K) = \kappa \geq w(K)$, we achieve even $K \in \mathcal{F}_f(\kappa)$ with the witnessing free dense subgroups each of cardinality $r(K)$. We shall see in Theorem 6.9 that the current hypothesis $r(K) \geq w(K)$ can be substantially weakened (to $r(K) \geq d(K)$) in case K is compact.

Theorem 6.5. *Let K be an infinite precompact abelian group such that $r(K) = \kappa \geq w(K) \geq \omega$. Then K admits an independent family $\{D_i : i \in I\}$ of free dense subgroups such that $|I| = \kappa$ and $|D_i| = r(K)$ for each $i \in I$.*

Proof. We first establish this statement.

(*) *If X is a subgroup of K with $|X| < \kappa$ and U is a nonempty open subset of K, then there exists non-torsion $s \in U$ such that $\langle s \rangle \cap X = \{0\}$.*

Let $A \subseteq K$ be an independent set such that $|A| = \kappa$. Since K is precompact there is finite $F \subseteq K$ such that $K = F + U$. Choose $z \in F$ such that $|(z + U) \cap A| = |A| = \kappa$, so that $|U \cap (A - z)| = |A| = \kappa$. Note that if $a_0, a_1 \in A$ satisfy $a_i - z \in t(K)$, say $n_i(a_i - z) = 0$ with $0 \neq n_i \in \mathbb{Z}$, then $(n_0 n_1)(a_0 - a_1) = 0$, contradicting the independence of A; hence, $|(A - z) \cap t(K)| \leq 1$, and we have that $A^* := (U \cap (A - z)) \setminus t(K)$ satisfies $|A^*| = |A| = \kappa$. If $\langle s \rangle \cap X \neq \{0\}$ for every $s \in A^*$, then for each such s there exists $n_x \in \mathbb{Z}$ such that $0 \neq n_x x \in X$, and then since $|X| < |\mathbb{Z} \times A^*|$, there are $n \in \mathbb{Z}$ and distinct $x, y \in A^*$ such that $n = n_x = n_y$; then $n(x - y) = 0$, contradicting the fact that A is independent. Hence, there is non-torsion $s \in A^* \subseteq U$ such that $\langle s \rangle \cap X = \{0\}$, as required, and (*) is proved.

Now let $\mathcal{T}^{\#}$ be a basis for K with $|\mathcal{T}^{\#}| = w(K)$. Finitely many translates of each $U \in \mathcal{T}^{\#}$ cover K, so $|U| = |K|$ for each such U.

We introduce two indexings of $\mathcal{T}^{\#}$: $\{U_i : i \in I\}$ is a faithful indexing of $\mathcal{T}^{\#}$ (with $|I| = w(K)$), and $\{U_\eta : \eta < \kappa\}$ is an indexing of $\mathcal{T}^{\#}$ in which each element of $\mathcal{T}^{\#}$ appears κ-many times. By recursion we will define for $\eta < \kappa$ subgroups X_η of K and points $s_\eta \in K$. Let $X_0 = \{0\}$ and, using (*), choose non-torsion $s_0 \in U_0$. Suppose now that $\eta < \kappa$ and that s_ξ and X_ξ have been defined for all $\xi < \eta$. Set $X_\eta := \langle \{s_\xi : \xi < \eta\} \rangle$. Then $|X_\eta| < \kappa$ so by (*) there is non-torsion $s_\eta \in U_\eta$ such that $\langle s_\eta \rangle \cap X_\eta = \{0\}$. Then X_η and s_η are defined for all $\eta < \kappa$. We set $S := \{s_\eta : \eta < \kappa\}$ and

$Y := \bigcup_{\eta<\kappa} X_\eta = \langle S\rangle$. Then S is an independent set, $\langle s_\eta\rangle \cong \mathbb{Z}$ for each $s_\eta \in S$, $|S \cap U_i| = \kappa$ for each $U_i \in \mathcal{T}^\#$, and

$$Y = \bigoplus_{s_\eta<|K|} \langle s_\eta\rangle \cong \bigoplus_{\kappa} \mathbb{Z}.$$

Now for $i \in I$, set $A_i := S \cap U_i$ and, as in the proof of Theorem 5.1 (this time with $\lambda := \kappa$), use the Disjoint Refinement Lemma to find pairwise disjoint sets $B_i : i \in I$ such that $|B_i| = \kappa$ and $B_i \subseteq A_i$ for each $i \in I$. Then with

$$D_i := \langle B_i\rangle = \bigoplus_{s\in B_i} \langle s\rangle \cong \bigoplus_{s\in B_i} \mathbb{Z} \cong \bigoplus_{\kappa} \mathbb{Z} \quad \text{for} \quad i \in I,$$

the family $\{D_i : i < \kappa\}$ is as required. □

According to Lemma 6.2(a), Theorem 6.5 provides a proof of Theorem D for all w-divisible compact abelian groups. We formulate this statement as a separate corollary.

Corollary 6.6. *If K is a w-divisible compact abelian group, then $K \in \mathcal{F}_f(|K|)$, i.e., K is maximally free-fragmentable.*

The following corollary is related as follows to the result quoted in section 1.3 from [37]: Unlike Corollary 6.7, that result excludes metrizable groups from consideration; further, the subgroups it yields are not free. But the subgroups given in [37] are not only dense in K, but even G_δ-dense.

Corollary 6.7. *If K is a compact connected abelian group, then K is maximally free-fragmentable, i.e., $K \in \mathcal{F}_f(|K|)$.*

6.1. Proofs of Theorem A and Theorem B.

We start with an application of Corollary 6.6.

Lemma 6.8. *Let K be a compact abelian group such that $K \notin \mathcal{F}(2)$. Then K is a torsion group.*

Proof. Suppose instead that $r(K) > 0$ and use Theorem 6.4 to write K in the form $K = K_{tor} \times K_w$, with K_{tor} a compact torsion group and with K_w compact and w-divisible. From $r(K) > 0$, we have $K_w \neq \{0\}$. According to Theorem 5.8, the compact torsion group K_{tor} splits $K_{tor} = K_{sing} \times K_{inf}$, with $K_{inf} \in \mathcal{F}(2)$ and $|K_{sing}| < \omega$.

The group $K_{sing} \times K_w$ is w-divisible, hence maximally free-fragmentable by Corollary 6.6, so $K_{sing} \times K_w \in \mathcal{F}(2)$. Then Lemma 4.1(b) gives

$$K = K_{tor} \times K_w = (K_{sing} \times K_{inf}) \times K_w = K_{inf} \times (K_{sing} \times K_w) \in \mathcal{F}(2). \quad \square$$

Now we are in position to prove Theorem A.

Proof of Theorem A. We are to prove that every compact abelian group K is in $\mathcal{F}_{ad}(2)$. If $K \in \mathcal{F}(2)$, there is nothing to prove, so (using Lemma 6.8) we assume that K is torsion. But then by Lemma 5.5 we have $K \in \mathcal{F}_{ad}(2)$, as required. □

Next we collect statements already proved, then we complete the proof of Theorem B.

Theorem 6.9. *For a compact abelian group K the following are equivalent:*

(i) $K \notin \mathcal{F}(2)$;
(ii) *K is torsion group, and* $K \notin \mathcal{F}(2)$;
(iii) *K is torsion and at least one of its leading Ulm-Kaplansky invariants is finite;*
(iv) $\mathbf{fin}(K) \neq \{0\}$.

Proof. The equivalence of (iii) and (iv) was proved in Lemma 2.13. The implication (i) $\Rightarrow$ (ii) follows from Lemma 6.8, while the implication (ii) $\Rightarrow$ (iv) follows from Lemma 5.5.

That (iv) $\Rightarrow$ (i) is clear, since from (iv) and Lemma 2.13(d), we have $\{0\} \neq \mathbf{fin}(K) \subseteq \mathbf{den}(K)$. □

Theorem 6.10. *Let K be an infinite compact abelian group. Then*

(a) $\mathbf{den}(K) \neq \{0\}$ *if and only if K is torsion with* $\mathbf{fin}(K) \neq \{0\}$ (*equivalently,* $K_{sing} \neq \{0\}$).
(b) $\mathbf{den}(K) = \mathbf{fin}(K)$; *moreover, there exist $D_1, D_2 \in \mathcal{D}(K)$ such that* $\mathbf{den}(K) = D_1 \cap D_2$;

Proof. (a) From Theorem 2.7 we have $\mathbf{fin}(K) \subseteq \mathbf{den}(K)$, so $\mathbf{den}(K) \neq \{0\}$ if $\mathbf{fin}(K) \neq \{0\}$. Conversely, if $\mathbf{den}(K) \neq \{0\}$, then $K \notin \mathcal{F}(2)$, so K is torsion with $\mathbf{fin}(K) \neq \{0\}$ by Theorem 6.9 and $K_{sing} \neq 0$ as well.

(b) From Theorem 2.7 we have the inclusion $\mathbf{fin}(K) \subseteq \mathbf{den}(K)$.

To show $\mathbf{den}(K) \subseteq \mathbf{fin}(K)$, recall first from Lemma 2.13 that $\mathbf{fin}(K/\mathbf{fin}(K)) = \{0\}$, so $K' := K/\mathbf{fin}(K) \in \mathcal{F}(2)$ by Theorem 6.9. Let $\pi : K \to K'$ be the quotient map and let $D_1', D_2' \in \mathcal{D}(K^+)$ witness the fact that $K' \in \mathcal{F}(2)$. By Corollary 4.7, $D_i := \pi^{-1}(D_i')$ $(i = 1, 2)$ is a pair of dense subgroups of K with $D_1 \cap D_2 = \ker\pi = \mathbf{fin}(K)$. This proves the inclusion $\mathbf{den}(K) \subseteq D_1 \cap D_2 = \mathbf{fin}(K)$. □

6.2. Proof of Theorem D.

To prove Theorem D, we need first the following strengthening of Theorem 6.5 (in the compact case), the condition $w(K) < r(K)$ there being here replaced by the weaker condition $d(K) \leq r(K)$. This may be compared also with Theorem 1.5, which under the much stronger assumption $r(K) = w(K)$ gave a similar conclusion for precompact groups.

We note also that the equivalence of (b) and (c) in Theorem 6.11 is obtained in a direct way in [27], based on more powerful results from [28], making no use of Theorem 6.4.

Theorem 6.11. *For a compact abelian group K, these conditions are equivalent:*

(a) *K is $r(K)$-free-fragmentable, i.e., $K \in \mathcal{F}_f(r(K))$;*
(b) *$K \in \mathcal{F}_f(1)$;*
(c) *$d(K) \leq r(K)$.*

Proof. (a) ⇒ (b) is obvious.

(b) ⇒ (c) There is a dense subgroup $F = \langle X \rangle \subseteq K$ with X free, and by Zorn's Lemma, there is a free set X' maximal with respect to the property $X \subseteq X' \subseteq K$, necessarily with $|X'| = r(K)$ ([32, Theorem 16.3], [35, Theorem (A.11)]), further with $r(K) \geq \mathfrak{c}$ by Theorem 1.4. Then with $F' := \langle X' \rangle$, we have $|F'| = |X'|$, and hence

$$d(K) \leq |F'| = |X'| = r(K),$$

as required.

(c) ⇒ (a) Write $K = B \times K_w$, where K_w is w-divisible and B is bounded. By Corollary 6.6 there exists an independent family of size $|K_w| = r(K)$ of dense free subgroups $\mathcal{D} = \{F_i : i < |K_w|\}$ of K_w, each of size $|K_w|$. Obviously, one can split this family in two disjoint families $\mathcal{D}' = \{F_i' : i < |K_w|\}$ and $\mathcal{D}'' = \{F_i'' : i < |K_w|\}$ of equal size ($\mathcal{D} = \mathcal{D}' \cup \mathcal{D}''$ is just an arbitrary partition in two parts of equal size). For each $i < |K_w|$, use the disjoint pair F_i' and F_i'' as above to build a dense free subgroup Γ_i of K as follows. Since $|F_i'| = r(K) \geq d(K) \geq d(B)$, there exists a homomorphism $\zeta_i : F_i' \to B$ with dense image. Extend ζ_i to $F = F_i' \oplus F_i''$ letting $\zeta_i \restriction_{F_i''} = 0$. Let Γ_i be the graph of ζ_i. Then Γ_i is a dense subgroup of $B \times K_w = K$ (as $F_i' \oplus F_i''$ is dense in K_w). Since obviously $\Gamma_i \cong F_i' \oplus F_i''$ is free, it remains only to note that the family $\{\Gamma_i : i < |K_w|\}$ is independent. □

We complete now the proof of Theorem D, recalling that Theorem 6.5 provides a proof for compact abelian groups K with $r(K) \geq w(K)$.

Proof of Theorem D. Consider now an arbitrary compact abelian group K such that $r(K) > 0$. By Lemma 6.3, $\mathfrak{f}(K) \leq r(K)$. Hence, it remains only to prove $\mathfrak{f}(K) \geq r(K)$.

By Theorem 6.4, K splits in a direct product $K = K_{tor} \times K_w$, where K_{tor} is a compact torsion group, while the compact group K_w is w-divisible. Let $\kappa = |K_w| = r(K_w) = 2^{w(K_w)} = r(K)$.

By Theorem 6.5, K_w is maximally free-fragmentable, i.e., $\mathfrak{f}(K_w) = \kappa$.

Decompose each $K_{tor}(p)$ $(p \in R(K_{tor}))$ into a product $K_{tor}(p) = M_p \times L_p$ so that in L_p all Ulm-Kaplansky invariants are $\leq \kappa$, while all Ulm-Kaplansky invariants in M_p are $> \kappa$ whenever $M_p \neq 0$. Then either $M := \prod_{p\in R(K)} M_p$ is zero, or

$$\mathfrak{f}(M) \geq \min\{\mathfrak{f}(M_p) : p \in R(G)\} \geq \kappa. \tag{6}$$

The product $L = \prod_{p\in\pi'} L_p$ has all Ulm-Kaplansky invariants $\leq \kappa$. In particular, $|L| \leq \kappa$. We show that $L \times K_w$ satisfies the hypothesis $r(L \times K_w) > w(L \times K_w)$ of Proposition 6.5. Indeed,

$$r(L \times K_w) = r(L) \cdot r(K_w) =$$

$$r(L) \cdot \kappa = |L| \cdot \kappa = 2^{w(L)} \cdot 2^{w(K_w)} > w(L) \cdot w(K_w) = w(L \times K_w).$$

Hence, $\mathfrak{f}(L \times K_w) = \kappa$, by Theorem 6.5. As

$$K = K_{tor} \times K_w = (M \times L) \times K_w = M \times (L \times K_w),$$

from (6) we deduce that $\mathfrak{f}(K) \geq \kappa$.

To conclude the proof, it remains to check the equivalence of (a)–(d). Conditions (a), (b), and (c) of Theorem D coincide with (a), (b), and (c), respectively, of Theorem 6.11. While (d) of Theorem D is obviously equivalent to (b) of Theorem D. □

It remains only to prove Corollary D2: For every cardinal κ, a compact abelian group $K \in \mathcal{F}_2(\kappa)$ satisfies $K \in \mathcal{F}(\kappa)$.

Proof of Corollary D2. If $K \in \mathcal{F}_2(\kappa)$ with $r(K) > 0$, then by Lemma 4.4 and Lemma 6.3 we have $r(K) \geq \kappa$, hence

$$K \in \mathcal{F}(r(K)) \subseteq \mathcal{F}(\kappa)$$

by Theorem D.

In case $K \in \mathcal{F}(2)$ is torsion, we have $\ell_{UK}(K) \geq \kappa$ from Lemma 4.5, and then Theorem C gives

$$K \in \mathcal{F}(\ell_{UK}(K)) \subseteq \mathcal{F}(\kappa).$$ □

As we pointed out in the Introduction (and later proved in Theorem D and Corollary D1), every compact abelian group K with $r(K) = |K|$ is both maximally fragmentable and maximally ad-fragmentable; i.e., $\mathfrak{f}(K) = \mathfrak{f}_2(K) = \mathfrak{f}_{ad}(K) = |K|$ (moreover, $\mathfrak{f}(K)$ can be added to this chain of equalities if $K \in \mathcal{F}(1)$).

This leaves open the question what happens in a compact abelian group K that is not maximally fragmentable. The torsion case was already faced by Question 4.6. Here we face the case $0 < r(K)$.

One has the following natural question.

Question 6.12. Let K be a compact abelian group with $0 < r(K)$. Does $K \in \mathcal{F}_{ad}(\kappa)$ imply $\kappa \leq r(K)$?

Since $\mathfrak{f}(K) = \mathfrak{f}_2(K) = r(K)$ by Theorem D, a positive answer to this question will prove that the almost disjoint fragmentation number $\mathfrak{f}_{ad}(K)$ exists and coincides with $\mathfrak{f}(K) = \mathfrak{f}_2(K) = r(G)$.

A negative answer to Question 6.12 opens various possibilities.

Question 6.13. Let K be a compact abelian group with $0 < r(K) < |K|$.

(a) Does the almost disjoint fragmentation number $\mathfrak{f}_{ad}(K)$ exist?
(b) If $\mathfrak{f}_{ad}(K)$ exists, is it equal to $r(K)$?
(c) If $\mathfrak{f}_{ad}(K)$ exists and $\mathfrak{f}_{ad}(K) > r(K)$, is the cardinal $\mathfrak{f}_{ad}(K)$ of exponential type?

In connection with Question 6.13, we know only that if $\mathfrak{f}_{ad}(K)$ exists, then $r(K) \leq \mathfrak{f}_{ad}(K) \leq |K|$.

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