

**CONTINUOUS EXTENSIONS OF FUNCTIONS  
DEFINED ON SUBSETS OF PRODUCTS  
WITH THE  $\kappa$ -BOX TOPOLOGY**

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**ABSTRACT.** Consider these results: (a) [N. Noble, Proc. Amer. Math.Soc. **31** (1972), 613–614] *every  $G_\delta$ -dense subspace in a product of separable metric spaces is  $C$ -embedded*; (b) [Milton Don Ulmer, Ph.D. Dissertation. Wesleyan University (Middletown, CT, USA), 1970 and Pacific J. Math. **46** (1973), 591–602] *every  $\Sigma$ -product in a product of first-countable spaces is  $C$ -embedded*; (c) [R. Pol and E. Puzio-Pol, Fund. Math. **93** (1976), no. 1, 57–69; also A. V. Arhangel'skii, Topology Proc. **25** (2000), Summer, 383–416, as corollaries of more general theorems] *every dense subset of a product of completely regular, first-countable spaces is  $C$ -embedded in its  $G_\delta$ -closure*.

The present paper continues the first two authors' earlier initiative [Topology Appl. **159** (2012), 2331–2337], which already generalized those cited results in several ways simultaneously, e.g.,  $\kappa$ -box topology on the product spaces; relaxed separation properties on both the domain and the range spaces). Now the authors show:

*Let  $\kappa \leq \alpha$  satisfy  $\lambda < \kappa$ ,  $\beta \leq \alpha \Rightarrow \beta^\lambda \leq \alpha$ ; let  $Y$  be dense in an open subset  $U$  of a  $\kappa$ -box product  $(\prod_{i \in I} X_i)_\kappa$  with each  $X_i$  a  $T_1$ -space; let  $q \in X_I \setminus Y$  have the property that for each  $J \in [I]^{\leq \alpha}$  there is  $y \in Y$  such that  $y_J = q_J$ ; let  $Z$  be a regular space with a  $\overline{G_{\alpha^+}}$ -diagonal. Suppose that for each  $i \in I$  either  $\chi(q_i, X_i) \leq \alpha$  or each intersection of  $\kappa$ -many neighborhoods of  $q_i$  is another such neighborhood. Then every continuous  $f : Y \rightarrow Z$  extends continuously over  $Y \cup \{q\}$ .*

Several corollaries and consequences are given.

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## 1. INTRODUCTION

Topological spaces considered here are not subjected to any standing separation properties. Additional hypotheses are imposed as required. Throughout this paper,  $\omega$  is the least infinite cardinal,  $\kappa$  and  $\alpha$  are infinite cardinals. For a set  $I$ , we define  $[I]^{<\kappa} := \{J \subseteq I : |J| < \kappa\}$ ; the symbol  $[I]^{\leq\kappa}$  is defined analogously. For a space  $X$  and  $x \in X$ , a set  $U \subseteq X$  is a *neighborhood* of  $x$  in  $X$  if  $x$  is in the interior of  $U$  in  $X$ . For  $A \subseteq X$ , we denote by  $\mathcal{N}_X(A)$ , or simply by  $\mathcal{N}(A)$  when ambiguity is unlikely, the set of open sets in  $X$  containing  $A$ . A point  $x \in X$  is a  $P(\kappa)$ -point of  $X$  if  $\bigcap \mathcal{V} \in \mathcal{N}(x)$  whenever  $\mathcal{V} \subseteq \mathcal{N}(x)$  and  $|\mathcal{V}| < \kappa$ ;  $X$  is a  $P(\kappa)$ -space provided each point  $x \in X$  is a  $P(\kappa)$ -point. Clearly, every topological space is a  $P(\omega)$ -space. The  $P(\omega^+)$ -spaces are often called  $P$ -spaces (see [11] and sources cited there).

For a space  $X$ ,  $x \in X$ , and  $A \subseteq X$ ,  $x$  belongs to the  $G_\kappa$ -closure of  $A$  in  $X$  if  $(\bigcap \mathcal{V}) \cap A \neq \emptyset$  whenever  $\mathcal{V} \subseteq \mathcal{N}(x)$  and  $|\mathcal{V}| < \kappa$ . A set  $A \subseteq X$  is a  $G_\kappa$ -set [a  $\overline{G_\kappa}$ -set, respectively] in  $X$  if there exists  $\mathcal{V} \subseteq \mathcal{N}_X(A)$  such that  $|\mathcal{V}| < \kappa$  and  $A = \bigcap \mathcal{V}$  [ $A = \bigcap \{\overline{V} : V \in \mathcal{V}\}$ , respectively]. (Thus, the familiar  $G_\delta$ -sets are exactly the  $G_{\omega^+}$ -sets.)  $V$  is a  $G_\kappa$ -neighborhood of  $x$  if there exists a  $G_\kappa$ -set  $U$  such that  $x \in U \subseteq V$ .

For spaces  $X$  and  $Z$  and  $Y \subseteq X$ , the symbol  $C(Y, Z)$  denotes the set of all continuous functions  $f : Y \rightarrow Z$ . The subspace  $Y$  of  $X$  is  $C(Z)$ -embedded in  $X$  provided each function  $f \in C(Y, Z)$  extends continuously over  $X$ ; when  $Z = \mathbb{R}$ ,  $Y$  is said to be  $C$ -embedded in  $X$ ; if  $Y$  is  $C(Z)$ -embedded in  $Y$  for every metrizable space  $Z$ , then  $Y$  is said to be  $\mathbf{M}$ -embedded in  $X$ .

Below we use the simple fact (which we will not mention again explicitly) that when  $Y$  is dense in  $X$  and  $Z$  is a regular  $T_1$ -space, a function  $f \in C(Y, Z)$  extends continuously over  $X$  if and only if  $f$  extends continuously to each point of  $X \setminus Y$  (restated:  $Y$  is  $C(Z)$ -embedded in  $X$  if and only if  $Y$  is  $C(Z)$ -embedded in  $Y \cup \{q\}$  for each  $q \in X \setminus Y$ ); in this connection, see [2] and [13].

For a set  $\{X_i : i \in I\}$  of sets and  $J \subseteq I$ , we write  $X_J := \prod_{i \in J} X_i$ ; and for every *generalized rectangle*  $A = \prod_{i \in I} A_i \subseteq X_I$ , the *restriction set* of  $A$ , denoted  $R(A)$ , is the set  $R(A) = \{i \in I : A_i \neq X_i\}$ . When each  $X_i = (X_i, \mathcal{T}_i)$  is a space, the symbol  $(X_I)_\kappa$  denotes  $X_I$  with the  $\kappa$ -box topology; this is the topology for which  $\{U : U = \prod_{i \in I} U_i, U_i \in \mathcal{T}_i, |R(U)| < \kappa\}$  is a base. Thus, the  $\omega$ -box topology on  $X_I$  is the usual product topology. We note that even when  $\kappa$  is regular, the intersection of fewer than  $\kappa$ -many sets, each open in  $(X_I)_\kappa$ , may fail to be open in  $(X_I)_\kappa$ .

A (not necessarily faithfully) indexed family  $\{A_i : i \in I\}$  of nonempty subsets of a space  $X$  is *locally*  $< \kappa$  if there is an open cover  $\mathcal{U}$  of  $X$  such that

$|\{i \in I : U \cap A_i \neq \emptyset\}| < \kappa$  for each  $U \in \mathcal{U}$ . A space  $X = (X, \mathcal{T})$  is *pseudo-* $(\alpha, \kappa)$ -*compact* if every indexed locally  $< \kappa$  family  $\{U_i : i \in I\} \subseteq \mathcal{T} \setminus \{\emptyset\}$  satisfies  $|I| < \alpha$ , and  $X$  is *pseudo-* $\alpha$ -*compact* if it is *pseudo-* $(\alpha, \omega)$ -*compact*. In this terminology, the familiar pseudocompact spaces are the pseudo- $\omega$ -compact spaces.

For  $p \in X_I = \prod_{i \in I} X_i$  and  $\alpha \geq \omega$ , the  $\alpha$ - $\Sigma$ -*product* of  $X_I$  based at  $p$  is the set  $\Sigma_\alpha(p) := \{x \in X_I : |\{i \in I : x_i \neq p_i\}| < \alpha\}$ . Usually  $\Sigma_\omega(p)$  is denoted by  $\sigma(p)$ .

The notation  $\kappa \ll \alpha$  means that  $\alpha$  is *strongly*  $\kappa$ -*inaccessible*. That is, (a)  $\kappa < \alpha$ , and (b)  $\beta^\lambda < \alpha$  whenever  $\beta < \alpha$  and  $\lambda < \kappa$  (see [8, p. 254]).

For (standard) topological definitions and notation not repeated here, see [9], [11], or [8].

**Remark 1.1.** The M.A. thesis [16], written by the third-listed co-author under the guidance of the second-listed co-author, incorporates many of the findings of this paper. The reader might consult it for commentary supplementing the historical background given in section 2 below.

## 2. HISTORICAL BACKGROUND. ULMER'S THEOREM

The point of departure for our work is the following theorem of Milton Don Ulmer [18], [17]. (We state his theorem using the notation introduced here.)

**Theorem 2.1** ([18, Theorem 2.2]). *Let  $\{X_i : i \in I\}$  be a set of Tychonoff spaces with  $p \in X_I$  and let  $\alpha \geq \omega$  be a cardinal. Suppose that either*

- (i)  $\Sigma_\alpha(p)$  is pseudo- $\alpha$ -compact, or
- (ii)  $\alpha$  is uncountable and  $x \in X_i$  implies  $\chi(x, X_i) < \alpha$ , or
- (iii)  $\alpha$  is uncountable and every  $X_i$  is a  $P$ -space.

*Then  $\Sigma_\alpha(p)$  is  $C$ -embedded in  $X_I$ .*

We note here that if  $\alpha = \omega$  in Theorem 2.1(i) so that  $\Sigma_\alpha(p) = \sigma(p)$ , then  $\sigma(p)$  must be a pseudocompact dense subset of  $X_I$ . But that cannot happen if  $|I| \geq \omega$  since  $\sigma(p)$  will not be a  $G_\delta$ -dense subset of  $X_I$ .

Theorem 2.1 has been generalized in several different ways over the years by many authors. Here we shall mention mainly those results that generalize Theorem 2.1 for the usual product topology and for the  $\kappa$ -box topology. We refer the reader to [12] and [8] and to the historical remarks contained in the other papers mentioned below for results not mentioned here.

(a) W. W. Comfort and S. Negrepontis in [7], and later in [8, Theorem 10.4], generalized the case of regular  $\alpha > \omega$  of (i) for the case of  $\kappa$ -box topology assuming that  $(X_I)_\kappa$  is a pseudo- $(\alpha, \kappa)$ -compact and  $\Sigma_\alpha(p)$  is

replaced by  $Y \subseteq X_I$  such that  $\pi_J[Y] = X_J$  for all nonempty  $J \in [I]^{<\alpha}$ ; and  $C$ -embedded is replaced by  $\mathbf{M}$ -embedded.

It must be noted here that the proofs in [8] depend on [8, Lemma 10.1], for which the proof was incomplete. In [6] the authors gave a complete proof of [8, Lemma 10.1] where no separation axioms were assumed for  $X_I$ . That allowed the authors to verify and unify all the results from [8, Chapter 10], whose status had become questionable, and to extend several of these. Later in [3] the results from [8] were generalized even further by replacing  $\mathbf{M}$ -embedded with  $C(Z)$ -embedded, where  $Z$  is a space such that the diagonal in  $Z$  is the intersection of  $< \alpha$ -many regular-closed subsets of  $Z \times Z$ . Those results generalized to the  $\kappa$ -box topology also some results from [12] obtained there for the usual product topology (for more specific information see [3]).

(b) The case  $\alpha = \omega^+$  of Theorem 2.1(ii) has been generalized in many different ways by many authors (see [4] for more information). In [5], Theorem 2.1(ii) was generalized for the case of  $\kappa$ -box topology and  $\alpha = \kappa^+$ , assuming that each  $x \in X_I$  is a  $P(\kappa)$ -point. In [4] the result from [5] and the case  $\alpha = \omega^+$  of Theorem 2.1(ii) were generalized in the following way. (For the definition when a subset  $Y$  of  $X_I$   $\alpha^+$ -duplicates a point  $q \in X_I$ , see Definition 3.2 below.)

**Theorem 2.2** ([4, Theorem 3.10]). *Let  $\omega \leq \kappa \leq \alpha$  with either  $\kappa < \alpha$  or  $\alpha$  regular, let  $\{X_i : i \in I\}$  be a set of  $T_1$ -spaces, and let  $Y$  be dense in an open subset of  $(X_I)_\kappa$ . Let  $q \in X_I \setminus Y$  be such that (i)  $Y$   $\alpha^+$ -duplicates  $q$  in  $X_I$  and (ii)  $q_i$  is a  $P(\alpha)$ -point in  $X_i$  with  $\chi(q_i, X_i) \leq \alpha$  for each  $i \in I$ . Then  $Y$  is  $C(Z)$ -embedded in  $Y \cup \{q\}$  for each regular space  $Z$  with a  $\overline{G_{\alpha^+}}$ -diagonal.*

(c) Theorem 2.2 does not generalize Theorem 2.1(iii); indeed, we are not aware of any published generalizations of Theorem 2.1(iii). Addressing that challenge, now in Theorem 3.10 and its corollaries (Theorem 3.11 and Theorem 3.12), we generalize Theorem 2.1(iii) in several directions simultaneously. We show further that in Theorem 2.2 the conjunction of the two hypotheses on the points  $q_i \in X_i$ , namely that each is a  $P(\alpha)$ -point and that each satisfies  $\chi(q_i, X_i) \leq \alpha$ , can be greatly relaxed: To achieve the desired conclusion, it is enough for each  $i \in I$  that  $q_i$  is a  $P(\kappa^+)$ -point of  $X_i$  or that  $\chi(q_i, X_i) \leq \alpha$ .

### 3. MAIN RESULTS

**Definition 3.1.** Let  $\kappa \geq \omega$  and let  $Z$  be a space. Then

- (a)  $\Delta(Z) := \{(z, z) \in Z \times Z : z \in Z\}$  is the *diagonal* of  $Z$  and
- (b)  $Z$  has a  $G_\kappa$ -*diagonal* [a  $\overline{G_\kappa}$ -*diagonal*, respectively] if  $\Delta(Z)$  is a  $G_\kappa$ -set [a  $\overline{G_\kappa}$ -set, respectively] in  $Z \times Z$ .

The terminology given in Definition 3.2(b) appears in [6, Definition 3.5].

**Definition 3.2.** Let  $\alpha \geq \omega$ , and let  $\{X_i : i \in I\}$  be a family of sets,  $Y \subseteq X_I := \prod_{i \in I} X_i$  and  $q \in X_I$ . Then

- (a) for  $\emptyset \neq J \subseteq I$ , we set  $Y_{q_J} := \{y \in Y : q_J = y_J\}$ , and
- (b)  $Y$   $\alpha$ -duplicates  $q$  if  $\emptyset \neq J \in [I]^{<\alpha} \Rightarrow Y_{q_J} \neq \emptyset$ .

**Remark 3.3.** (a) The knowledgeable reader will have speculated (correctly) that the authors of [4] introduced the concept of  $\alpha$ -duplication as a useful generalization of the familiar  $\Sigma_\alpha$ -products and subspaces  $Y \subseteq X_I$  such that  $\pi_J[Y] = X_J$  for all nonempty  $J \in [I]^{<\alpha}$ .

(b) To help fix ideas further, we note that if  $Y \subseteq X_I$  with each  $X_i$  a discrete space, then  $Y \subseteq X_I$  is dense in  $(X_I)_\alpha$  if and only if  $Y$   $\alpha$ -duplicates each  $q \in X_I$ . More generally we have, as remarked in [4], if  $\kappa \leq \alpha^+$  and  $q$  is a point in  $X_I$  such that  $\psi(q_i, X_i) \leq \alpha$  for each  $i \in I$ , then  $Y$   $\alpha^+$ -duplicates  $q$  in  $X_I$  if and only if  $q$  belongs to the  $G_{\alpha^+}$ -closure of  $Y$  in  $(X_I)_\kappa$ .

(c) If, in (b), the condition  $\psi(q_i, X_i) \leq \alpha$  is replaced by the condition that each  $q_i$  is a  $P(\kappa^+)$ -point of  $X_i$  (see Theorem 3.10(i) below), then, even when  $\alpha = \kappa = \omega$ , a  $G_{\alpha^+}$ -dense subset  $Y$  of a space  $(X_I)_\kappa$  need not  $\alpha^+$ -duplicate every point of  $(X_I)_\kappa$ . For an example, it is enough to take  $X = D \cup \{q\}$  with  $D$  discrete,  $|D| = \omega^+$ , neighborhoods of  $q$  are co-countable, and  $Y := D^{\omega^+} \subseteq X^{\omega^+} = (X^{\omega^+})_\omega$ .

For what follows we need the following result.

**Theorem 3.4** ([4, Theorem 3.8]). *Let  $\omega \leq \kappa \leq \alpha$  with either  $\kappa < \alpha$  or  $\alpha$  regular, let  $\{X_i : i \in I\}$  be a set of spaces, and let  $Y$  be dense in an open subset  $U$  of  $(X_I)_\kappa$ . Let  $q \in X_I \setminus Y$  be a point such that  $Y$   $\alpha^+$ -duplicates  $q$ , and let  $Z$  be a space with a  $\overline{G_{\alpha^+}}$ -diagonal. Then, for every  $f \in C(Y, Z)$ , there are  $S \in [I]^{<\alpha}$  and  $z \in Z$  such that  $f(y) = z$  whenever  $y \in Y$  satisfies  $y_S = q_S$ .*

**Definition 3.5.** Let  $X, Y$ , and  $Z$  be spaces,  $Y \subseteq X$ ,  $V \subseteq Z$ , and  $f : Y \rightarrow Z$ . Then

- (a)  $W \subset X$  is  $V$ -small (for  $f$ ) if  $f[W \cap Y] \subseteq V$ , and
- (b) if  $y \in Y$ ,  $W \in \mathcal{N}_X(y)$ , and  $W$  is  $V$ -small, then  $W$  is a  $V$ -neighborhood of  $y$ .

**Definition 3.6.** Let  $\omega \leq \kappa \leq \alpha$ , let  $\{X_i : i \in I\}$  be a set of spaces, and let  $Y$  be a subset of  $(X_I)_\kappa$ . Let  $q \in X_I$ ,  $V$  be open in a space  $Z$ , and  $f \in C(Y, Z)$ . Then  $J \subseteq I$  is  $V_\alpha$ -cofinal for  $q$  if (i)  $|J| < \alpha$  and (ii) for every  $J' \in [I \setminus J]^{<\alpha}$ , there exist  $y \in Y_{q_{J'}}$  and a basic open  $V$ -neighborhood  $W$  of  $y$  in  $(X_I)_\kappa$  such that  $R(W) \cap J' = \emptyset$ .

We acknowledge in greater detail our substantial structural and intellectual debt to Ulmer [18], whose Theorem 2.2 is generalized below in our Theorem 3.10 and its corollaries, Theorem 3.11 and Theorem 3.12. First, our Definition 3.6 (of a  $V_\alpha$ -cofinal set  $J \subseteq I$ ) closely parallels, and is motivated by, his Definition 2.5 of an  $\epsilon_\gamma$ -cofinal set  $J \subseteq I$  when  $\alpha = \aleph_\gamma$ ; our Lemma 3.7, showing the existence for  $V \in \mathcal{N}_Z(z)$  of a special  $V_{\alpha+}$ -cofinal set  $J_V \in [I]^{\leq \alpha}$  for  $q$ , parallels his Lemma 2.6; and our surprising Lemma 3.8, showing that there is a  $V_{\alpha+}$ -cofinal set  $K_V \subseteq I$  for  $q$  of cardinality  $< \kappa$ , precisely parallels his construction (in his more limited context) of a finite  $\epsilon_\gamma$ -cofinal set  $J \subseteq I$ .

Though our new concepts and results, then, clearly parallel Ulmer's, there are substantial differences. Strictly considered, our concept of a  $V_{\alpha+}$ -cofinal set does not formally generalize Ulmer's concept of an  $\epsilon_\gamma$ -cofinal set since in our case we define the concept for every point  $q \in X_I$  (and we use this concept only for points  $q \notin Y$ ), while Ulmer (when  $Y = \Sigma_\alpha(p) = \Sigma_{\aleph_\gamma}(p)$  and  $Z = \mathbb{R}$ ) restricts to the case  $q \in Y$ . When  $q \notin Y$ , then  $f(q)$  is not defined and that is why in our definition we need a point  $y$  from  $Y_{q,J}$ . But then, even when our Definition 3.6 is altered to allow only the case  $q \in Y = \Sigma_\alpha(p)$  and  $Z = \mathbb{R}$ , that definition and Ulmer's with  $\alpha = \aleph_\gamma$  are still different because of the involvement of the set  $Y_{q,J}$ .

**Lemma 3.7.** *Let  $\kappa, \alpha, X_I, Y, Z, U, q, f, S$ , and  $z$  be as in Theorem 3.4. Then for every  $V \in \mathcal{N}_Z(z)$ , there exists  $J_V \in [I]^{\leq \alpha}$  such that  $S \subseteq J_V$  and  $J_V$  is  $V_{\alpha+}$ -cofinal for  $q$ .*

*Proof.* Suppose that for some  $V \in \mathcal{N}_Z(z)$  no such  $J_V$  exists. Recursively, for ordinals  $\eta \leq \alpha$ , we will define sets  $S_\eta \in [I]^{\leq \alpha}$  with  $S_\eta \supseteq S_\xi$  for all  $\xi < \eta$ .

Set  $S_0 := S$  and suppose for  $\eta < \alpha$  that  $S_\xi \in [I]^{\leq \alpha}$  has been defined for all  $\xi \leq \eta$ , with  $S_{\xi'} \subseteq S_\xi \subseteq S_\eta$  whenever  $\xi' < \xi \leq \eta$ . Since  $S_\eta$  is not  $V_{\alpha+}$ -cofinal for  $q$ , there is  $J'_\eta \in [I \setminus S_\eta]^{\leq \alpha}$  such that for every  $y \in Y_{q,S_\eta}$  and for every  $V$ -neighborhood  $W$  of  $y$  in  $(X_I)_\kappa$ , we have  $R(W) \cap J'_\eta \neq \emptyset$ . We set  $S_{\eta+1} := S_\eta \cup J'_\eta$ .

For limit ordinals  $\eta \leq \alpha$ , we set  $S_\eta := \bigcup_{\xi < \eta} S_\xi$ .

The recursive definitions are complete. We note that  $|S_\alpha| \leq \alpha$ .

Now, let  $y \in Y_{q,S_\alpha}$  and let  $W$  be a basic open  $V$ -neighborhood of  $y$  in  $(X_I)_\kappa$ . Since  $|R(W)| < \kappa$  and  $\kappa < \alpha$  or  $\alpha$  is regular, there exists  $\eta_0 < \alpha$  such that  $R(W) \cap (S_{\eta_0+1} \setminus S_{\eta_0}) = \emptyset$  for each  $\eta$  with  $\eta_0 < \eta < \alpha$ . But for each such  $\eta$ , we have  $y \in Y_{q,S_\eta}$ , and hence  $R(W) \cap (S_{\eta+1} \setminus S_\eta) \neq \emptyset$ , a contradiction.  $\square$

**Lemma 3.8.** *Let  $\omega \leq \kappa \ll \alpha^+$ , let  $\{X_i : i \in I\}$  be a set of spaces, and let  $Y$  be dense in an open subset  $U$  of  $(X_I)_\kappa$ . Let also  $q \in X_I \setminus Y$  be a point such that  $Y$   $\alpha^+$ -duplicates  $q$ , and let  $Z$  be a space with a  $G_{\alpha^+}$ -diagonal. Let  $f \in C(Y, Z)$  and let  $S$  and  $z$  be as given by Theorem 3.4:  $S \in [I]^{\leq \alpha}$ , and  $f(y) = z$  for all  $y \in Y$  such that  $y_S = q_S$ . Then for every  $V \in \mathcal{N}_Z(z)$ , there exists  $K_V \in [I]^{< \kappa}$  such that  $K_V$  is  $V_{\alpha^+}$ -cofinal for  $q$  and  $K_V \subseteq J_V$ , where  $J_V$  is as given by Lemma 3.7.*

*Proof.* Given  $V$ , let  $J_V$  be as in Lemma 3.7:  $J_V \in [I]^{\leq \alpha}$  and  $J_V$  is  $V_{\alpha^+}$ -cofinal for  $q$ . If there exists  $y \in Y_{q, J_V}$  and a basic  $V$ -neighborhood  $W$  of  $y$  in  $(X_I)_\kappa$  with  $R(W) \subseteq J_V$ , then clearly  $K_V := R(W)$  will be as required. Therefore, in what follows, we suppose that for every  $y \in Y_{q, J_V}$  and every basic  $V$ -neighborhood  $W$  of  $y$  in  $(X_I)_\kappa$ , we have  $R(W) \setminus J_V \neq \emptyset$ .

Let  $y_0 \in Y_{q, J_V}$  and let  $W_0$  be a basic  $V$ -neighborhood of  $y_0$  in  $(X_I)_\kappa$ . Then  $R(W_0) \setminus J_V \neq \emptyset$ . Since  $J_V$  is  $V_{\alpha^+}$ -cofinal for  $q$ , there exists a point  $y_1 \in Y_{q, J_V}$  and a basic  $V$ -neighborhood  $W_1$  of  $y_1$  in  $(X_I)_\kappa$  with  $R(W_1) \cap (R(W_0) \setminus J_V) = \emptyset$ . According to our assumption,  $R(W_1) \setminus J_V \neq \emptyset$ , and therefore  $R(W_1) \setminus (R(W_0) \cup J_V) \neq \emptyset$ . We continue recursively: For  $\eta < \alpha^+$ , we choose  $y_\eta \in Y_{q, J_V}$  and a basic open  $V$ -neighborhood  $W_\eta$  of  $y_\eta$  such that  $R(W_\eta) \cap (\bigcup_{\xi < \eta} R(W_\xi) \setminus J_V) = \emptyset$ . Then, for every  $\xi < \eta < \alpha^+$ , we have  $R(W_\xi) \cap R(W_\eta) \subseteq J_V$ . Since  $|R(W_\eta)| < \kappa$  and  $\kappa \ll \alpha^+$ , it follows from the Erdős-Rado theorem for quasi-disjoint families (see [10] or [8, Theorem 1.4]) that there are a set  $A \subseteq \alpha^+$  with  $|A| = \alpha^+$  and a set  $K_V \in [I]^{< \kappa}$  such that  $R(W_\xi) \cap R(W_\eta) = K_V \subseteq J_V$  for every  $\xi, \eta \in A$ ,  $\xi \neq \eta$ .

To see that  $K_V$  is  $V_{\alpha^+}$ -cofinal for  $q$ , let  $K \in [I]^{\leq \alpha}$  satisfy  $K_V \cap K = \emptyset$ . Since  $\{R(W_\eta) \setminus K_V : \eta \in A\}$  is a collection of  $\alpha^+$ -many pairwise disjoint subsets of  $I$ , there exists an index  $\eta_1 \in A$  such that  $R(W_{\eta_1}) \cap K = \emptyset$ . Then  $y_{\eta_1}$  and  $W_{\eta_1}$  are as required.  $\square$

**Remark 3.9.** We note that in Lemma 3.8 the possibility  $K_V = \emptyset$  is not excluded. We claim in that case that if  $Z$  is regular, then the function  $\bar{f} : Y \cup \{q\} \rightarrow Z$  defined by

$$\bar{f}|_Y = f, \quad \bar{f}(q) = z$$

satisfies  $\bar{f}[Y \cup \{q\}] \subseteq V$ , so that every neighborhood of  $q$  in  $Y \cup \{q\}$  is a  $V$ -neighborhood.

To verify that, let  $V' \in \mathcal{N}_Z(z)$  be such that  $\bar{V}' \subseteq V$  and let  $\{R(W'_\eta) : \eta \in A\}$  be a family constructed as in the proof of Lemma 3.8 (now for the set  $V'$  in place of  $V$ ): This is a family of  $\alpha^+$ -many pairwise disjoint subsets of  $I \setminus J_{V'}$ , with each  $W'_\eta$  a basic open  $V'$ -neighborhood of some  $y_\eta \in Y_{q, J_{V'}} \subseteq U$ . Without loss of generality we can assume that  $W'_\eta \subseteq U$  for each  $\eta \in A$ . Now suppose that there exists  $y \in Y$  such that  $f(y) \notin V$ .

Since  $f$  is continuous at  $y$ , there exists a basic open neighborhood  $W \subseteq U$  of  $y$  such that  $f[W] \subseteq Z \setminus \overline{V'}$ . Since  $|R(W)| < \kappa$ , there is  $\eta \in A$  such that  $R(W) \cap R(W'_\eta) = \emptyset$ . Then  $\emptyset \neq W \cap W'_\eta \subset U$ , and since  $Y$  is dense in  $U$ , there is  $y_0 \in Y \cap W \cap W'_\eta$ . Then from  $y_0 \in W'_\eta$ , we have  $f(y_0) \in V'$ , a contradiction since from  $y_0 \in W$  we have also  $f(y_0) \in Z \setminus \overline{V'}$ .

**Theorem 3.10.** *Let  $\omega \leq \kappa \ll \alpha^+$ , let  $\{X_i : i \in I\}$  be a set of  $T_1$ -spaces, and let  $Y$  be dense in an open subset  $U$  of  $(X_I)_\kappa$ . Let  $q \in X_I \setminus Y$  be a point such that  $Y$   $\alpha^+$ -duplicates  $q$ , and let  $Z$  be a regular space with a  $\overline{G_{\alpha^+}}$ -diagonal. Suppose for each  $i \in I$  that either  $q_i$  is a  $P(\kappa^+)$ -point of  $X_i$ , or  $\chi(q_i, X_i) \leq \alpha$ . Then  $Y$  is  $C(Z)$ -embedded in  $Y \cup \{q\}$ .*

*Proof.* Let  $f \in C(Y, Z)$ . Since  $Y$   $\alpha^+$ -duplicates  $q$  in  $X_I$ , we have from Theorem 3.4 that there exist  $z \in Z$  and  $J \in [I]^{\leq \alpha}$  such that  $f(y) = z$  for all  $y \in Y$  satisfying  $y_J = q_J$ . We define  $\overline{f} : Y \cup \{q\} \rightarrow Z$  by the rule

$$\overline{f}|_Y = f, \quad \overline{f}(q) = z.$$

We must show  $\overline{f} \in C(Y \cup \{q\}, Z)$ . Since  $Y$  is open in  $Y \cup \{q\}$ , the function  $\overline{f}$  remains continuous at each  $y \in Y$ . To show that  $\overline{f}$  is continuous at  $q$ , we show that for each  $V' \in \mathcal{N}_Z(z)$ , there is a  $V'$ -neighborhood  $U'$  of  $q$ .

Since  $Z$  is regular, there is  $V \in \mathcal{N}_Z(z)$  such that  $\overline{V} \subseteq V'$ . Let  $J_V \in [I]^{\leq \alpha}$  be as in Lemma 3.7:  $S \subseteq J_V$  and  $J_V$  is  $V_{\alpha^+}$ -cofinal for  $q$ . Let also  $K_V \subseteq J_V$  be as in Lemma 3.8, i.e.,  $K_V \in [I]^{< \kappa}$  and  $K_V$  is  $V_{\alpha^+}$ -cofinal for  $q$ .

If  $K_V = \emptyset$ , then, according to Remark 3.9, the set  $Y \cup \{q\}$  is a  $V'$ -neighborhood of  $q$  in  $Y \cup \{q\}$ , as required. If there exists a  $V$ -neighborhood  $W$  of some  $y \in Y_{q_{J_V}}$  such that  $R(W) \subseteq J_V$ , then we are done. If not, then, as in the proof of Lemma 3.8, we choose recursively a transfinite sequence  $\{y_\eta : \eta < \alpha^+\}$  of points such that  $y_\eta \in Y_{q_{J_V}}$  and a transfinite sequence  $\{W_\eta : \eta < \alpha^+\}$ , where each  $W_\eta$  is a basic open  $V$ -neighborhood of  $y_\eta$  in  $(X_I)_\kappa$  with the property that  $R(W_\xi) \cap R(W_\eta) = K_V$  whenever  $\xi < \eta < \kappa$ .

For each  $\eta < \alpha^+$ , the set  $\widetilde{W}_\eta := \prod_{i \in K_V} (W_\eta)_i$  is a neighborhood of  $q_{K_V}$  in  $(X_{K_V})_\kappa$ .

We will define the required  $V'$ -neighborhood  $U'$  of  $q$  in  $(X_I)_\kappa$  with the help of the following definition:

$$K_\chi := \{i \in K_V : \chi(q_i, X_i) \leq \alpha\}.$$

If  $K_\chi = \emptyset$ , we set  $B := \bigcap_{\eta < \kappa} \widetilde{W}_\eta = \prod_{i \in K_V} (\bigcap_{\eta < \kappa} (W_\eta)_i)$ . Since  $q_i$  is a  $P(\kappa^+)$ -point of  $X_i$  for each  $i \in K_V$ , we have that  $B$  is a neighborhood in  $(X_{K_V})_\kappa$  of  $q_{K_V}$ .

If  $K_\chi \neq \emptyset$ , we claim that  $\chi(q_{K_\chi}, (X_{K_\chi})_\kappa) \leq \alpha$ . Indeed, for  $i \in K_\chi$ , let  $\{(B_\eta)_i : \eta < \alpha\}$  be a base at  $q_i$  in  $X_i$ , and for  $\phi \in \alpha^{K_\chi}$ , let  $B(\phi) :=$



$\prod_{i \in K_\chi} (B_{\phi(i)})_i$ . Then  $\mathcal{B} := \{B(\phi) : \phi \in \alpha^{K_\chi}\}$  is a base at  $q_{K_\chi}$  in  $(X_{K_\chi})_\kappa$ , and  $|\mathcal{B}| = \alpha$ , since, from  $|K_\chi| < \kappa \ll \alpha^+$ , we have  $|\alpha^{K_\chi}| = \alpha$ . Therefore, for each  $\eta < \alpha^+$ , there is  $B(\phi) \in \mathcal{B}$  such that  $B(\phi) \subseteq (\widetilde{W}_\eta)_{K_\chi}$ . It follows that there exist (fixed)  $B_\chi \in \mathcal{B}$  and  $\Lambda \in [\alpha^+]^{\alpha^+}$  such that  $B_\chi \subseteq (\widetilde{W}_\eta)_{K_\chi}$  for each  $\eta \in \Lambda$ . Now choose  $\Lambda' \in [\Lambda]^\kappa$  and set  $B := \bigcap_{\eta \in \Lambda'} \widetilde{W}_\eta = \prod_{i \in K_V} (\bigcap_{\eta \in \Lambda'} (W_\eta)_i)$ . Since  $q_i$  is a  $P(\kappa^+)$ -point of  $X_i$  for each  $i \in K_V \setminus K_\chi$  and  $B_\chi \subseteq (\widetilde{W}_\eta)_{K_\chi}$  for each  $\eta \in \Lambda'$ , we have again that  $B$  is a neighborhood in  $(X_{K_V})_\kappa$  of  $q_{K_V}$ .

The set  $B$  has been defined in each case. It remains to show that  $U' := B \times \prod_{i \in I \setminus K_V} X_i$  is a  $V'$ -neighborhood of  $q$  in  $(X_I)_\kappa$ .

If not, then there exists a point  $y \in Y \cap U'$  such that  $f(y) \notin V'$ . Then  $f(y) \in Z \setminus \overline{V}$  and since  $f$  is continuous at  $y$ , we can find a basic  $(Z \setminus \overline{V})$ -neighborhood  $W'$  of  $y$  with  $W' \subseteq U$ . Note that when  $K_\chi = \emptyset$ , the family  $\{R(W_\eta) \setminus K_V : \eta < \kappa\}$  is a family of  $\kappa$ -many pairwise disjoint subsets of  $I$ ; and when  $K_\chi \neq \emptyset$ , the family  $\{R(W_\eta) \setminus K_V : \eta \in \Lambda'\}$  is such a family. Then, since  $|R(W')| < \kappa$ , there is  $\eta_0$  (with  $\eta_0 < \kappa$  when  $K_\chi = \emptyset$ ;  $\eta_0 \in \Lambda'$  otherwise) such that  $(R(W_{\eta_0}) \setminus K_V) \cap R(W') = \emptyset$ , and hence  $R(W_{\eta_0}) \cap R(W') \subseteq K_V$ . Thus,  $W_{\eta_0} \cap W' \neq \emptyset$ , and since  $Y$  is dense in  $U$  and  $W' \subseteq U$ , we have  $Y \cap W_{\eta_0} \cap W' \neq \emptyset$ . Then, with  $y' \in Y \cap W_{\eta_0} \cap W'$ , we have this contradiction:  $f(y') \in Z \setminus \overline{V}$  since  $y' \in W'$ , and  $f(y') \in V$  since  $y' \in W_{\eta_0}$ .  $\square$

We note two consequences of Theorem 3.10 and Remark 3.3(b).

**Theorem 3.11.** *Let  $\omega \leq \kappa \ll \alpha^+$ , let  $\{X_i : i \in I\}$  be a set of  $T_1$ -spaces, and let  $Y$  be dense in an open subset  $U$  of  $(X_I)_\kappa$ . Let  $q \in X_I \setminus Y$  be a point in the  $G_{\alpha^+}$ -closure of  $Y$  in  $(X_I)_\kappa$ , and let  $Z$  be a regular space with a  $\overline{G_{\alpha^+}}$ -diagonal. If  $\chi(q_i, X_i) \leq \alpha$  for each  $i \in I$ , then  $Y$  is  $C(Z)$ -embedded in  $Y \cup \{q\}$ .*

**Theorem 3.12.** *Let  $\omega \leq \kappa \ll \alpha^+$ , let  $\{X_i : i \in I\}$  be a set of  $T_1$ -spaces, let  $Y$  be dense in an open subset  $U$  of  $(X_I)_\kappa$ , and let  $Z$  be a regular space with a  $\overline{G_{\alpha^+}}$ -diagonal. If every  $q \in X_I \setminus Y$  in the  $G_{\alpha^+}$ -closure of  $Y$  in  $(X_I)_\kappa$  satisfies  $\chi(q_i, X_i) \leq \alpha$  for each  $i \in I$ , then  $Y$  is  $C(Z)$ -embedded in its  $G_{\alpha^+}$ -closure in  $(X_I)_\kappa$ .*

**Remark 3.13.** (a) Suppose for a moment that in our principal result, Theorem 3.10, the hypothesis is (unnecessarily) strengthened to require that either

- (i)  $q_i$  is a  $P(\kappa^+)$ -point of  $X_i$ , for each  $i \in I$ , or
- (ii)  $\chi(q_i, X_i) \leq \alpha$  for each  $i \in I$ .

Then, when  $\kappa = \omega$  and  $Z = \mathbb{R}$  and  $Y$  is a  $\Sigma_{\alpha^+}$ -product in  $X_I$ , the resulting two theorems slightly generalize [18, Theorem 2.2.(iii) and 2.2.(ii)],

respectively. Similarly in this case, Theorem 3.12 slightly generalizes [18, Theorem 2.2.(ii)].

(b) In the same vein, we note that Theorem 3.11, also the case when  $\chi(q_i, X_i) \leq \alpha$  for each  $i \in I$  of Theorem 3.10, compares with [4, Theorem 3.10] as follows: The conclusions are the same, the hypothesis  $\omega \leq \kappa \leq \alpha$  with either  $\kappa < \alpha$  or  $\alpha$  regular is strengthened to  $\omega \leq \kappa \ll \alpha^+$ , and the hypothesis that  $q_i$  is a  $P(\alpha)$ -point in  $X_i$  for each  $i \in I$  is eliminated.

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