Topological Combinatorics: A Peaceful Pursuit

W. W. Comfort

The diversion of science to destructive purposes is as old as science itself. The observation that a club, knife, spear or arrow useful to bring down an animal could be used to similar effect against "the enemy" in the next village was readily perverted to the design of weapons strictly for anti-personnel purposes. Even while enhancing the human condition, science has degraded and brutalized it—early on with the misappropriation of potentially neutral or useful discoveries (gunpowder, the airplane) and more recently and egregiously by the development of malevolent technologies dedicated exclusively to killing: rockets, the H-bomb, biological warfare. The history of science is replete with examples of bright intellects in three categories: those who, driven either by a savage spirit or by ambition, lead and participate willingly in the project to kill; those who participate minimally and without enthusiasm; and those who refuse, often at personal risk. (For a carefully researched, illuminating account of the actions of German mathematicians in the first half of the twentieth century, see [11]. I know of no parallel comparably detailed study of a significant sector of the scientific community in the United States, where pressures to participate were much less strong and penalties for defiance commensurably less severe.)

The Irish number theorist Henry J. S. Smith is said [10] to have proposed this toast, perhaps on the occasion (1874) of his inauguration as President of the London Mathematical Society: "Here's to Mathematics—may she never be of use to anybody." In contrast to Smith, who evidently recognized at least the possibility of the (mis)use of mathematics, G. H. Hardy [5] was able, remarkably, to write in 1940 that "real mathematics has no effects on war . . . Mathematics is . . . a harmless and innocent occupation." Perhaps one may excuse this inexcusable pronouncement on the grounds that Hardy, surely a "real mathematician" by any workable standard, had somehow satisfied himself that the pertinent moral and ethical issues could be safely declared irrelevant, in effect defined out of existence. To him, applicable mathematics is disjoint from the "real"; indeed, "real mathematics must be justified as art if it can be justified at all" [5].

If the knowable universe it not quite Euclidean, surely it is metrizable and σ-compact, hence separable (and Hausdorff). This puts an upper bound (perhaps $\mathfrak{c}$, or $2^{\mathfrak{c}}$ to be safe) on the set of cardinal numbers which may reasonably be associated with any real-world problem. Thus a set-theoretic topologist working with cardinal invariants is almost surely living a professional life consistent with the familiar dictum of Hippocrates (which does not, incidentally, appear in his famous Oath) "First, do no harm." As a goal in itself, to "do no harm" seems pitifully unproductive and defensive, but as a point of departure it is an admirable counsel. To an emerging scholar approaching mathematical pubescence and seeking a rich and challenging outlet for emerging creative energies, I can think of no path less susceptible to deflection into destructive purposes than infinitary combinatorics. (For centuries, prime numbers were viewed deservedly as the purest of the pure. But Number Theory lost its innocence some decades ago with the advent of computer-driven encryption.)

Dept. of Math., Wesleyan University, Middletown, CT 06459, USA; E-mail: wcomfort@wesleyan.edu

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The price of clean hands is admittedly nontrivial. According to the logic here being advanced, the cost of leaving the world uncorrupted is to leave it unimproved.

It would stretch the truth beyond credibility to assert that set-theoretic topology *is* combinatorics. But examples abound which establish the suggestion that combinatorics is the bedrock upon which much of topology lies. I select here six combinatorial principles (abbreviated **CP**), all in ZFC, with in each case one or more topological consequences (**TC**). I use standard topological notation as given in the references cited, as well as these conventions. Always η is an ordinal, α and κ are infinite cardinals, and $U_\kappa(\alpha)$ is the set of κ-uniform ultrafilters on α, that is, those $p \in \beta(\alpha)$ for which $A \in p \Rightarrow |A| \geq \kappa$; we write $U(\alpha) := U_\alpha(\alpha)$. For a space $X = (X, \mathcal{T})$, $S(X)$ is the *Souslin number* of X, that is, the least cardinal not the cardinality of a cellular family in X; and X_α is X with the topology generated by $\{\cap\mathcal{U} : \mathcal{U} \in [\mathcal{T}]^{<\alpha}\}$. For a set $\{X_i : i \in I\}$ of spaces, X_I denotes $\Pi_i X_i$ with the usual product topology.

Must an ω-indexed family $\{A_n : n < \omega\}$ of finite sets have an infinite subfamily whose pairwise intersections coincide? Surely not. It is enough to take $A_n = n = \{0, 1, \cdots, n-1\}$; here, no three sets have pairwise identical intersection. That trivial example introduces and lends interest to the following combinatorial principle.

CP1 [Erdős and Rado] (1960, 1969). If $[\beta < \alpha$ and $\lambda < \kappa] \Rightarrow \beta^\lambda < \alpha$, then for every α-indexed family $\{A_\eta : \eta < \alpha\}$ of sets of cardinality $< \kappa$ there is a subfamily $\{A_\eta : \eta \in A\}$ with $A \in [\alpha]^\alpha$ whose pairwise intersections coincide; and conversely.

TC1 $S(X_I) = \sup\{S(X_F) : F \in [I]^{<\omega}\}$.

CP2 [Erdős and Rado] (1956), [Kurepa] (1959). $(\beth_n(\alpha))^+ \to (\alpha^+)^{n+1}_\alpha$; in particular, $(2^\alpha)^+ \to (\alpha^+)^2_\alpha$.

TC2.1 If each $S(X_i) \leq \alpha^+$, then $S(X_F) \leq (2^\alpha)^+$ $(|F| < \omega)$, and hence $S(X_I) \leq (2^\alpha)^+$ by **TC1**.

So in particular, the anomalous behavior with respect to cellularity of a Souslin Line L is "bounded". One has $\omega^+ = S(L) < S(L \times L)$, but $\mathfrak{c}^+ = (2^\omega)^+$ is a universal upper bound for all numbers of the form $S(X_0 \times X_1)$ where the spaces X_i satisfy the countable chain condition. The world of mathematics is sometimes hectic, but it is not totally out of control.

TC2.2 A compact space X with $S(X) \leq \alpha^+$ satisfies $S(X_{\alpha^+}) \leq (2^\alpha)^+$.

Similarly here, upon adjoining all "$G_{\leq\alpha}$" sets to the topology of X, the "Souslin jump" is limited to a single exponentiation.

The remarkable and unexpected improvements of **TC2.1** and **TC2.2** achieved by Negrepontis and his school in the period 1980 ± 2, using combinatorial generalizations of **CP1** and **CP2**, seem inadequately known and appreciated: When $Y = X_I$ (or even $Y = (X_I)_{\alpha^+}$) in **TC2.1** or $Y = X_{\alpha^+}$ in **TC2.2**, then not only is $S(Y) \leq (2^\alpha)^+$ but $(2^\alpha)^+$ is a pre-calibre for Y. That means: Not only is no $(2^\alpha)^+$-indexed family $\{U_\eta : \eta < (2^\alpha)^+\}$ of nonempty open subsets of Y a pairwise disjoint family, but each such indexed family has a subfamily $\{U_\eta : \eta \in A\}$ with $|A| = (2^\alpha)^+$ with the finite intersection property.

We remark in passing that although a compact topological group G necessarily satisfies $S(G) \leq \omega^+$, this question of van Douwen apparently remains unsolved: Is there an upper bound on numbers of the form $S(X)$ with X a compact, homogeneous space? Is $\mathfrak{c}^+$ such a number?

CP3 Given $|S| = 2^\alpha$, there is $D \subseteq \alpha^S$ with $|D| = \alpha$ such that: for each $f \in \alpha^S$ and $F \in [S]^{<\omega}$ there is $g \in D$ such that $g|F = f|F$.

Here **CP3** is, of course, a statement in combinatorial language of the topological fact that the product of 2^α-many copies of the discrete space α has a dense subset of cardinality α. This is turn yields the Hewitt-Marczewski-Pondiczery theorem:

TC3.1 If $|I| \leq 2^\alpha$ and each $d(X_i) \leq \alpha$ ($i \in I$), then $d(X_I) \leq \alpha$.

The relation between **CP3** and topology is symbiotic, in that **CP3** itself is provable from elementary topological properties of the space $S = \{0, 1\}^\alpha$. There follows in turn a familiar topological corollary to **TC3.1**.

TC3.2 [Marczewski] (1941) for $\alpha = \omega$, X_i metrizable; [Shanin] (1946). If each $d(X_i) \leq \alpha$ ($|I|$ arbitrary), then $S(X_I) \leq \alpha^+$. Hence each $f \in C(X_I, \mathbb{R})$ factors as $f = g \circ \pi_J$ for suitable $J \in [I]^{\leq\alpha}$, $g \in C(X_J, \mathbb{R})$.

The special case of **TC3.1** asserting that $[0, 1]^\mathfrak{c}$ (alternatively, $\mathbb{R}^\mathfrak{c}$) is separable, always a surprise to undergraduates, is rendered transparent by Mrówka [8] *via* the observation that there are (only) countably many polynomials with rational coefficients. **TC3.3** with $\alpha = \omega$ then follows. Similarly, Oxtoby [9] contributed a nifty *ad hoc* proof that $S(X_I) \leq \omega^+$ when each X_i is separable: One easily associates with each $X_i = (X_i, \mathcal{T}_i)$ a probability measure μ_i such that $\mu_i(U) > 0$ whenever $\emptyset \neq U \in \mathcal{T}_i$, and since the product measure assigns positive measure to each basic open set in X_I, each cellular family there is countable. My efforts to find a generalization of either of these special arguments to the full statements given in **TC3.1** and **TC3.2** respectively have been unsuccessful.

When Čech (1937) described the fundamental properties of $\beta(X)$, whose existence he generously attributed to Tychonoff (1929/30), he left the cardinal number $|\beta(\alpha)|$ undetermined. Since **TC3.1** shows that $\beta(\alpha)$ maps continuously onto $\{0, 1\}^{2^\alpha}$, **CP3** gives the answer.

TC3.3 [Hausdorff] (1936), [Pospíšil] (1937). $|\beta(\alpha)| = 2^{2^\alpha}$.

CP4 [Erdős] (1934). Given $|S| = \alpha^{<\kappa} := \Sigma_{\lambda<\kappa}\alpha^\lambda$, there is $\mathcal{A} \in [\mathcal{P}(S)]^{\alpha^\kappa}$ such that $A_0 \neq A_1$ (in $\mathcal{A}$) implies $|A_0 \cap A_1| < \kappa$. In particular [Tarski] (1928), there is $\mathcal{A} \in [\mathcal{P}(\alpha)]^{\alpha^\omega}$ such that $A_0 \neq A_1$ (in $\mathcal{A}$) implies $|A_0 \cap A_1| < \omega$.

TC4 $S(\beta(\alpha)\backslash\alpha) = (\alpha^\omega)^+$; $S(U_\kappa(\alpha^{<\kappa})) = (\alpha^\kappa)^+$; $S(U(\alpha)) = (2^\alpha)^+$ when $\alpha = 2^{<\alpha}$; and $S(U_{\alpha^+}(2^\alpha)) = (2^{(\alpha^+)})^+$.

In the following principle, a set $\mathcal{A}$ chosen maximal with respect to the relevant property is shown, by an argument reminiscent of Cantor's famous diagonalization proof that $|\mathbb{R}| > \omega$, not to have cardinality α. Choosing $\mathcal{A}$ so that $|\mathcal{A}| \geq \alpha$, one concludes that $|\mathcal{A}| > \alpha$, a strict inequality reflected in the strict inequalities of **TC5.1** and **TC5.2**. In the absence of additional hypotheses, for example GCH, the more precise (and desirable) result $|\mathcal{A}| = 2^\alpha$ in **CP5** is not available. Indeed Baumgartner [1] gives a model in which every $\mathcal{A}$ as in **CP5** (with $\alpha = \omega^+$) has $|\mathcal{A}| < 2^{(\omega^+)}$ (and hence $S(U(\omega^+)) \leq 2^{(\omega^+)}$).

CP5 There is $\mathcal{A} \in [\alpha^\alpha]^{\alpha^+}$ such that $f \neq g$ (in $\mathcal{A}$) implies $|\{\eta \in \alpha : f(\eta) = g(\eta)\}| < \alpha$. Equivalently: there is $\mathcal{A} \in [\mathcal{P}(\alpha)]^{\alpha^+}$ such that $A_0 \neq A_1$ (in $\mathcal{A}$) implies $|A_0 \cap A_1| < \alpha$.

TC5.1 $S(U(\alpha)) > \alpha^+$.

TC5.2 [Frayne, Morel and Scott] (1962) For $p \in U(\alpha)$, the ultrapower α^α/p satisfies $|\alpha^\alpha/p| > \alpha$.

A principle related to **CP5**, due to Sierpiński (1949) and developed in [1] and with many topological consequences, is that the partially ordered set $(\mathcal{P}(2^{<\alpha}), \subseteq)$ contains a chain of length 2^α; hence $\mathcal{P}(\alpha)$ contains a chain of length α^+.

CP6 (The Disjoint Refinement Lemma). For each α-indexed family $\{A_\eta : \eta < \alpha\}$ of sets, with each $|A_\eta| = \alpha$, there are pairwise disjoint sets $B_\eta \in [A_\eta]^\alpha$.

Proving **CP6** is a nice training exercise in well-ordering and the use of ordinals at the introductory graduate level. As with **CP1**, an appealing feature of the principle is its triviality at the extremes: When the given sets A_η coincide or are pairwise disjoint. The applications of **CP6** touch many topological subdisciplines. For a space $X = (X, \mathcal{T})$ write $\Delta(X) = \min\{|U| : \emptyset \neq U \in \mathcal{T}\}$, and recall that X is *maximally resolvable* if it has $\Delta(X)$-many pairwise disjoint dense subsets.

TC6.1 (Ceder [2]) If $\pi w(X) \leq \Delta(X)$, then X is maximally resolvable.

TC6.2 (a) $|U(\alpha)| = 2^{2^\alpha}$, and (b) $p \in U(\alpha) \Rightarrow \chi(p, U(\alpha)) > \alpha$.

Another appeal to **CP3** strengthens **TC6.2**(a):

TC6.3 [Pospíšil] (1939) $|\{p \in U(\alpha) : \chi(p, U(\alpha)) = 2^\alpha\}| = 2^{2^\alpha}$; indeed [Juhász] (1969) $|\{p \in U(\alpha) : \chi(p, U(\alpha)) < 2^\alpha\}| < 2^{2^\alpha}$.

Summary. Many aspects of general (set-theoretic) topology relating to cardinal invariants are rooted in infinitary combinatorics. In a world where benign science is too easily turned to malign purposes, this direction of inquiry is recommended as a safe haven, invitingly free of such applications.

Bibliographic Remarks. **CP1** and **CP2** are discussed and proved in full, together with the consequences mentioned here and many others, in [6] and [7]. Somewhat the same material is addressed in [3] and [4], as are **CP3–6** and a wealth of other combinatorial principles and topological consequences. The bibliographic citations given in truncated form above are available in [3] and [4] in the familiar extended format.

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