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M-embedded subspaces of certain product spaces

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Abstract

A subspace Y of a space X is said to be **M**-*embedded* in X if every continuous $f : Y \to Z$ with Z metrizable extends to a continuous function $\overline{f} : X \to Z$.

For topological spaces X_i $(i \in I)$ and $J \subseteq I$, set $X_J := \prod_{i \in J} X_i$.

The authors prove a general theorem concerning κ-box topologies and pseudo-(α, κ)-compact spaces, of which the following is a corollary of the special case $\kappa = \alpha = \omega$.

Theorem. *If $Y \subseteq X_I$ and $\pi_J[Y] = X_J$ for all $\emptyset \neq J \in [I]^{<\omega^+}$, and if each X_J, for $\emptyset \neq J \in [I]^{<\omega}$, is Lindelöf, then Y is* **M**-*embedded in X_I.*

Remark. Several results in Chapter 10 of the book [W.W. Comfort, S. Negrepontis, Chain Conditions in Topology, Cambridge Tracts in Math., vol. 79, Cambridge Univ. Press, 1982] depend on Lemma 10.1, of which the given proof was incomplete. A principal contribution here is to furnish a correct proof, allowing the present authors to verify and unify all the results from Chapter 10 whose status had become questionable, and to extend several of these.

MSC: primary 54B10, 54D35; secondary 54A35

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1. Introduction and historical perspective

Conventions, Notation, Definitions 1.1. (a) Topological spaces considered here are not subjected to any special standing separation properties. Specific hypotheses are imposed as required.

(b) ω is the least infinite cardinal, and α and κ are infinite cardinals. For I a set and β an arbitrary cardinal we write $[I]^\beta := \{J \subseteq I\colon |J| = \beta\}$; the notations $[I]^{<\beta}$, $[I]^{\leqslant\beta}$ are defined analogously.

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(c) A (not necessarily faithfully) indexed family $\{A_i\colon i \in I\}$ of nonempty subsets of a space X is *locally* $< \kappa$ if there is an open cover $\mathcal{U}$ of X such that $|\{i \in I\colon U \cap A_i \neq \emptyset\}| < \kappa$ for each $U \in \mathcal{U}$. A space $X = (X, \mathcal{T})$ is *pseudo-*(α, κ)*-compact* if every indexed locally $< \kappa$ family $\{U_i\colon i \in I\} \subseteq \mathcal{T}\backslash\{\emptyset\}$ satisfies $|I| < \alpha$, and X is *pseudo-*α*-compact* if it is *pseudo-*(α, ω)*-compact*. In this terminology, the familiar pseudocompact spaces are the pseudo-ω-compact spaces.

(d) For a set $\{X_i\colon i \in I\}$ of sets and $J \subseteq I$, we write $X_J := \prod_{i\in J} X_i$; and for $A = \prod_{i\in I} A_i \subseteq X_I$ the *restriction set of* A, denoted $R(A)$, is the set $R(A) = \{i \in I\colon A_i \neq X_i\}$. When each $X_i = (X_i, \mathcal{T}_i)$ is a space, the symbol $(X_I)_\kappa$ denotes X_I with the κ*-box topology*; this is the topology for which $\{U = \prod_{i\in I} U_i\colon U_i \in \mathcal{T}_i,\ |R(U)| < \kappa\}$ is a base. Thus the ω-box topology on X_I is the usual product topology. We note that even when κ is regular, the intersection of fewer than κ-many sets, each open in $(X_I)_\kappa$, may fail to be open in $(X_I)_\kappa$.

(e) The symbol $\mathbb{R}$ denotes the real line with its usual topology. For spaces Y and Z we denote by $C(Y, Z)$ the set of continuous functions from Y into Z. A subspace Y of a space X is C^*-embedded [respectively C*-embedded*; respectively $\mathbf{M}$*-embedded*] if every $f \in C(Y, [0, 1])$ [respectively $f \in C(Y, \mathbb{R})$; respectively $f \in C(Y, M)$ with M metrizable] extends to $\overline{f} \in C(X, [0, 1])$ [respectively $\overline{f} \in C(X, \mathbb{R})$; respectively $\overline{f} \in C(X, M)$].

In the interest of symmetry and efficiency, we say that a space is *compact Hausdorff* [respectively *realcompact*; respectively *topologically complete*] if it is homeomorphic to a closed subspace of a space of the form $[0, 1]^\kappa$ [respectively $\mathbb{R}^\kappa$; respectively $\prod_{i\in I} M_i$ with each M_i metrizable]. Evidently, each such space is a Tychonoff space. Given a Tychonoff space X, the symbols $\beta(X)$, $\upsilon(X)$, and $\gamma(X)$ denote, respectively, the Stone–Čech compactification, the Hewitt realcompactification, and the Dieudonné topological completion, of X; we have

$$\upsilon(X) = \{p \in \beta(X)\colon X \text{ is } C\text{-embedded in } X \cup \{p\}\}, \quad \text{and}$$
$$\gamma(X) = \{p \in \beta(X)\colon X \text{ is } \mathbf{M}\text{-embedded in } X \cup \{p\}\}.$$

The space $\beta(X)$ [respectively $\upsilon(X)$; respectively $\gamma(X)$] is, up to a homeomorphism leaving X fixed pointwise, the unique compact Hausdorff [respectively realcompact; respectively topologically complete] space in which X is dense and C^*-embedded [respectively C-embedded; respectively $\mathbf{M}$-embedded].

We refer the reader to [14] for a thorough development and treatment of the spaces $\beta(X)$, $\upsilon(X)$, and $\gamma(X)$. See also [6,7].

Social Background 1.2. As a graduate student reading background material preparatory for [27], Luis Recoder-Núñez in 1999 noted a gap or *non-sequitur* in the proof of Lemma 10.1 of [7]. (See Lemma 2.1 below for a *verbatim* restatement of that lemma.) This event cast into doubt many of the subsequent theorems in Chapter 10 of [7]. More recently, other mathematicians observed the same error in the proof of that lemma, but left unsettled the question of its truth. Subsequently the authors of [7] received two purported counterexamples to Lemma 10.1, but these also did not withstand close scrutiny: In one, both the hypothesis and the conclusion of Lemma 10.1 were satisfied; in the other, both failed.

In Section 2 of the present paper we give a careful proof of Lemma 10.1 in a form which is more general than that of in [7], thus validating the assertions of [7, Chapter 10]. Additional considerations in Section 3 allow us to extend to the class of topologically complete spaces certain results presented in [7] only for realcompact spaces.

2. Lemma 10.1 from [7]

We begin with a restatement and a correct proof of the lemma in question, without assuming any separation axioms for the spaces X_i, $i \in I$ (we note that in [7] all spaces are assumed completely regular). In order that this paper be self-contained, we give the proof in full detail, retaining insofar as is convenient the notation and the format of the argument from [7]. In Remark 2.3(c) we specify the location and the nature of the error in [7].

Lemma 2.1. *Let $\omega \leqslant \kappa \leqslant \alpha$ with either $\kappa < \alpha$ or α regular, let $\{X_i\colon i \in I\}$ be a set of non-empty spaces, Y a dense, pseudo-(α, κ)-compact subspace of $(X_I)_\kappa$, (M, ρ) a metric space and f a continuous function from Y to M. Then for every $\epsilon > 0$ there is $J \in [I]^{<\alpha}$ such that $\rho(f(x), f(y)) \leqslant \epsilon$ if $x, y \in Y$, $x_J = y_J$.*

Proof. We suppose the result fails.

For $\xi < \alpha$ we will define $x(\xi), y(\xi) \in Y$, basic neighborhoods $U(\xi)$ and $V(\xi)$ in $(X_I)_\kappa$ of $x(\xi)$ and $y(\xi)$, respectively and $J(\xi), A(\xi) \subseteq I$ such that:

(i) $\rho(f(x), f(y)) > \epsilon$ if $x \in U(\xi) \cap Y$, $y \in V(\xi) \cap Y$;
(ii) $A(\xi) := \{i \in R(U(\xi)) \cup R(V(\xi))\colon\ x(\xi)_i \neq y(\xi)_i\}$;
(iii) $U(\xi)_i = V(\xi)_i$ if $i \in I \backslash A(\xi)$;
(iv) $x(\xi)_i = y(\xi)_i$ for $i \in J(\xi)$; and further with
(v) $J(0) = \emptyset$, $J(\xi) = \bigcup_{\eta<\xi} A(\eta)$ for $0 < \xi < \alpha$.

To begin, we choose $x(0) \in Y$ and $y(0) \in Y$ such that $\rho(f(x(0)), f(y(0))) > \epsilon$. It follows from the continuity of f that there are disjoint, basic open neighborhoods $\widetilde{U(0)}$ and $\widetilde{V(0)}$ in $(X_I)_\kappa$ of $x(0)$ and $y(0)$, respectively such that $\rho(f(x), f(y)) > \epsilon$ for all $x \in \widetilde{U(0)} \cap Y$, $y \in \widetilde{V(0)} \cap Y$. Then, define $A(0) := \{i \in R(\widetilde{U(0)}) \cup R(\widetilde{V(0)})\colon\ x(0)_i \neq y(0)_i\}$ and define (basic) open neighborhoods $U(0)$ and $V(0)$ in $(X_I)_\kappa$ of $x(0)$ and $y(0)$, respectively, as follows:

$$U(0)_i = V(0)_i = X_i \quad \text{if } i \in I \backslash \big(R\big(\widetilde{U(0)}\big) \cup R\big(\widetilde{V(0)}\big)\big);$$
$$U(0)_i = V(0)_i = \widetilde{U(0)}_i \cap \widetilde{V(0)}_i \quad \text{if } i \in \big(R\big(\widetilde{U(0)}\big) \cup R\big(\widetilde{V(0)}\big)\big) \backslash A(0); \quad \text{and}$$
$$U(0)_i = \widetilde{U(0)}_i, \qquad V(0)_i = \widetilde{V(0)}_i \quad \text{if } i \in A(0).$$

Then (i)–(v) hold for $\xi = 0$.

Suppose now that $0 < \xi < \alpha$ and that $x(\eta), y(\eta) \in Y$, $U(\eta)$, $V(\eta)$, and $A(\eta), J(\eta) \subseteq I$ have been defined for $\eta < \xi$ satisfying (the analogues of) (i)–(v). Since $J(\xi)$, defined by (v), satisfies $|J(\xi)| < \alpha$, there are $x(\xi)$ and $y(\xi)$ in Y such that (iv) holds and $\rho(f(x(\xi)), f(y(\xi))) > \epsilon$. It follows from the continuity of f that there are disjoint, basic open neighborhoods $\widetilde{U(\xi)}$ and $\widetilde{V(\xi)}$ in $(X_I)_\kappa$ of $x(\xi)$ and $y(\xi)$, respectively, such that $\rho(f(x), f(y)) > \epsilon$ for all $x \in \widetilde{U(\xi)} \cap Y$, $y \in \widetilde{V(\xi)} \cap Y$. Then, define $A(\xi) := \{i \in R(\widetilde{U(\xi)}) \cup R(\widetilde{V(\xi)})\colon\ x(\xi)_i \neq y(\xi)_i\}$ and define (basic) open neighborhoods $U(\xi)$ and $V(\xi)$ in $(X_I)_\kappa$ of $x(\xi)$ and $y(\xi)$, respectively, as follows:

$$U(\xi)_i = V(\xi)_i = X_i \quad \text{if } i \in I \backslash \big(R\big(\widetilde{U(\xi)}\big) \cup R\big(\widetilde{V(\xi)}\big)\big);$$
$$U(\xi)_i = V(\xi)_i = \widetilde{U(\xi)}_i \cap \widetilde{V(\xi)}_i \quad \text{if } i \in \big(R\big(\widetilde{U(\xi)}\big) \cup R\big(\widetilde{V(\xi)}\big)\big) \backslash A(\xi); \quad \text{and}$$
$$U(\xi)_i = \widetilde{U(\xi)}_i, \qquad V(\xi)_i = \widetilde{V(\xi)}_i \quad \text{if } i \in A(\xi).$$

Then (i)–(v) hold. The recursive definitions are complete.

We note that if $\eta < \xi < \alpha$ and $i \in A(\eta)$ then $x(\xi)_i = y(\xi)_i$ and hence $i \notin A(\xi)$. That is: the sets $A(\xi)$ $(\xi < \alpha)$ are pairwise disjoint.

The space Y is pseudo-(α, κ)-compact, hence there is $\overline{p} \in Y$ such that each basic open neighborhood W of $\overline{p}$ in $(X_I)_\kappa$ satisfies $|\{\xi < \alpha\colon\ W \cap U(\xi) \neq \emptyset\}| \geqslant \kappa$. Fix such W and choose $\bar{\xi} < \alpha$ such that $W \cap U(\bar{\xi}) \neq \emptyset$ and no $i \in R(W)$ is in $A(\bar{\xi})$. (This is possible since $|R(W)| < \kappa$ and each $i \in R(W)$ is in at most one of the sets $A(\xi)$.) For each such $\bar{\xi}$ by (iii) we have $U(\bar{\xi})_i = V(\bar{\xi})_i$ for all $i \in R(W)$, so also $W \cap V(\bar{\xi}) \neq \emptyset$.

Since Y is dense in $(X_I)_\kappa$, the previous paragraph shows this: For each neighborhood W in $(X_I)_\kappa$ of $\overline{p}$ there is $\bar{\xi}$ such that $W \cap U(\bar{\xi}) \cap Y \neq \emptyset$ and $W \cap V(\bar{\xi}) \cap Y \neq \emptyset$. From (i) it then follows that the oscillation of f on every neighborhood in Y of $\overline{p}$ is more than ϵ; hence f is not continuous at $\overline{p} \in Y$. This contradiction completes the proof. □

In Remarks 2.3 and Theorem 2.4 we use some properties of generalized Σ-products to expose the inadequacies of the argument proposed in [7] to prove Lemma 2.1. The reader interested only in topological consequences of that lemma may skip directly to Section 3.

Notation 2.2. For $p \in X_I = \prod_{i \in I} X_i$, the *$\kappa$-$\Sigma$-product of X_I based at p* is the set $\Sigma_\kappa(p) := \{x \in X_I\colon\ |\{i \in I\colon\ x_i \neq p_i\}| < \kappa\}$.

Remarks 2.3. (a) In Notation 2.2, the usual Σ-product based at p is the set $\Sigma(p) = \Sigma_{\omega^+}(p)$, and the "little σ-product" [8] is the set $\sigma(p) := \Sigma_\omega(p) \subseteq X_I$. If $|X_i| \geqslant 2$ for all $i \in I$ then $\pi_J[\Sigma_\kappa(p)] = X_J$ iff $J \in [I]^{<\kappa}$, so if the sets X_i are topological spaces then each $\Sigma_\kappa(p) \subseteq X_I$ is dense in $(X_I)_\kappa$.

(b) So far as we are aware, Σ-products were introduced and first studied by Mazur [22]. Corson [8] showed for separable metric spaces X_i that each Σ-product $\Sigma(p) \subseteq X_I$ is (dense and) C-embedded in X_I. (It is clear in this case if in addition each X_i is compact, then $\Sigma(p)$ is pseudocompact.) Corson's theorems, and related theorems of Glicksberg [15] and Kister [21], have been substantially extended. It was shown, for example, that a G_δ-dense subspace Y of a space X is C-embedded in X provided

(i) (Noble [23]) $X = X_I$ with each X_i separable metric; or
(ii) (Pol and Pol [25]) $X = X_I$ with each X_i first-countable; or
(iii) (Hernández and Sanchis [17]) X is a compact topological group.

The most incisive tool used in the last quarter-century to study the C-embedded property of certain (dense) subspaces is Arhangel'skiĭ's concept of a Moscow space [1]: A Hausdorff space is *Moscow* if the closure of each of its open subsets is the union of G_δ-subsets. Arhangel'skiĭ has shown [1–4] that the class of Moscow spaces is very broad, including (among many others) spaces X with any of these properties: X is a product of first countable spaces; X is a locally bounded topological group; X is dense in some Moscow space. (Some of those assertions depend in part on work of Yajima [32].) In 1989 Uspenskii [30], using work of Tkačenko [28], showed that every G_δ-dense subspace of a Moscow space is C-embedded. The culmination of this line of investigation is the following result of Arhangel'skiĭ [4]: *A space is Moscow if and only if each of its dense subspaces is C-embedded in its own G_δ-closure.* This remarkable theorem subsumes and unifies many earlier results—including, for example, those cited above in (i)–(iii).

(c) The relevance of (a) to Lemma 2.1 above (which is Lemma 10.1 of [7]) is readily discerned upon specializing there to the case $\kappa = \alpha = \omega$. In this case the lemma asserts, correctly, that for $\epsilon > 0$ and for every continuous $f : Y \to M$ with Y a dense, pseudocompact subspace of a space X_I and M metrizable, there is finite $J \subseteq I$ such that $\rho(f(x), f(y)) \leqslant \epsilon$ whenever $x, y \in Y$, $x_J = y_J$. But the proof in [7] suggests, incorrectly, that for arbitrary $\widetilde{x}, \widetilde{y} \in Y$ and finite $J \subseteq I$ there are x and y in Y such that $x_J = \widetilde{x}_J$, $y_J = \widetilde{y}_J$, and $x \in \sigma(y)$. That is correct in case Y contains a σ-product—the case which has historically motivated much of this inquiry—but it can fail in general. We have not seen an example of that possible failure given explicitly in the literature, so we offer one now.

Theorem 2.4. *Let $\{X_i : i \in I\}$ be a (not necessarily faithfully indexed) set of compact metric spaces with each $|X_i| > 1$ and with $|I| = \kappa > \omega$. There is a dense, pseudocompact subspace Y of X_I such that $|Y \cap \sigma(p)| \leqslant 1$ for each $p \in X_I$.*

Proof. In fact we show more: Y may be chosen so that even $|Y \cap \Sigma_\kappa(p)| \leqslant 1$ for each $p \in X_I$.

As indicated above, a dense set $Y \subseteq X_I$ is pseudocompact iff Y is G_δ-dense in X_I. Every nonempty G_δ-set in X_I contains, for some countable $C \subseteq I$ and some $p \in X_C$, a set of the form $\{p\} \times X_{I\setminus C}$. There are κ^ω possible choices for C, and $\mathfrak{c}$-many choices for $p \in X_C$. Thus, there are κ^ω-many "basic G_δ" subsets of X_I. We index these as $\{U_\eta : \eta < \kappa^\omega\}$, and we note that each U_η meets each set of the form $\Sigma(r) \subseteq \Sigma_\kappa(r) \subseteq X_I$.

Now for $i \in I$ choose distinct $p_i, q_i \in X_i$. Let $\{I_\xi : \xi < \kappa\}$ be a partition of κ with each $|I_\xi| = \kappa$, and for $A \subseteq \kappa$ define $r(A) \in X_I$ by

$$r(A)_i = \begin{cases} p_i & \text{if } i \in I_\xi,\ \xi \in A, \\ q_i & \text{if } i \in I_\xi,\ \xi \in \kappa\setminus A. \end{cases}$$

If A and B are distinct subsets of κ then $r(A)_i \neq r(B)_i$ for all $i \in I_\xi$ with $\xi \in (A\setminus B) \cup (B\setminus A) \neq \emptyset$, so $r(B) \notin \Sigma_\kappa(r(A))$ and hence $\Sigma_\kappa(r(A)) \cap \Sigma_\kappa(r(B)) = \emptyset$.

Finally, let $\{A(\eta) : \eta < \kappa^\omega\}$ be (distinct) subsets of κ and for $\eta < \kappa^\omega$ choose $y(\eta) \in U_\eta \cap \Sigma_\kappa(r(A_\eta))$. The set $Y := \{y(\eta) : \eta < \kappa^\omega\}$ is G_δ-dense in X_I, and $|Y \cap \Sigma_\kappa(r)| \leqslant 1$ for each κ-Σ-product $\Sigma_\kappa(r) \subseteq X_I$. □

3. Some topological consequences

The rest of this paper is devoted to deriving some consequences—some new, some already accessible in [7]—of Lemma 2.1. The following useful definition is strictly set-theoretic, in the sense that topology plays no role in its statement. For its applications, of course, X_i and Z will be spaces, and $f \in C(Y, Z)$.

Definition 3.1. Let $f : Y \to Z$, with $Y \subseteq X_I$.

(a) If $J \subseteq I$, then f *depends on* J if $[x, y \in Y$ and $x_J = y_J] \Rightarrow f(x) = f(y)$; and
(b) f depends on $< \alpha$*-many* [respectively $\leqslant \alpha$*-many*] *coordinates* if there is $J \in [I]^{<\alpha}$ [respectively $J \in [I]^{\leqslant\alpha}$] such that f depends on J.

We remark that if α, κ, X_i, Y, M and f are as in Lemma 2.1, then f depends on $\leqslant \alpha$-many coordinates; if in addition $\mathrm{cf}(\alpha) > \omega$ then f depends on $< \alpha$-many coordinates. [The proof is obvious: For $0 < n < \omega$ there is $J_n \in [I]^{<\alpha}$ such that $\rho(f(x), f(y)) \leqslant \frac{1}{n}$ if $x, y \in Y$ with $x_{J_n} = y_{J_n}$, so f depends on $J := \bigcup_{0<n<\omega} J_n$.]

In the following proposition, of which versions have been noted by many authors in differing contexts (see for example [7, 10.3]), no separation properties whatever are imposed on any of the hypothesized spaces. We include a proof in the interest of completeness.

Theorem 3.2. *Let $\kappa \leqslant \alpha$, Y be a subspace of $(X_I)_\kappa$ such that $\pi_J[Y] = X_J$ whenever $\emptyset \neq J \in [I]^{<\alpha}$, and let $f \in C(Y, Z)$ depend on $< \alpha$-many coordinates. Then f extends to a continuous function $\overline{f} : (X_I)_\kappa \to Z$.*

Proof. Let f depend on $\emptyset \neq J \in [I]^{<\alpha}$. For $p \in X_I$ choose $y \in Y$ such that $p_J = y_J$, and define $\overline{f} : X_I \to Z$ by $\overline{f}(p) = f(y)$. Since $\pi_{J'}[U] = \pi_{J'}[U \cap Y]$ for each basic open set U of $(X_I)_\kappa$ and for each set $J' \in [I]^{<\alpha}$, the continuity of $\overline{f}$ on $(X_I)_\kappa$ follows from the continuity of f on Y. □

Remark 3.3. Theorem 3.2 is a simple result, with little depth. We pause for a moment to reflect upon a qualitative distinction in flavor between Theorem 3.2 and those deeper theorems of General Topology which, for Y dense in some space X and for some space(s) Z, guarantee that every $f \in C(Y, Z)$ extends to $\overline{f} \in C(X, Z)$ (e.g., the extension theorem of Stone and Čech, Lavrentieff's theorem, and so forth). Typically there, range($\overline{f}$) properly contains range(f), and $\overline{f}$ is defined at the points of $X \backslash Y$ using some sort of convergence property, or an argument of Baire category type, for Z. In Theorem 3.2, in contrast, the hypotheses on the disposition or placement of Y within X are sufficiently strong that each point $p \in X = X_I$ associates naturally with a point $y \in Y$ such that $p_J = y_J$, and the natural definition $\overline{f}(p) = f(y)$ renders unnecessary the consideration of any completeness properties which Z may enjoy (and ensures that range($\overline{f}$) = range(f)).

Discussion 3.4. Lemma 2.1 applies when one has in hand a pseudo-(α, κ)-compact space which is dense in a space of the form $(X_I)_\kappa$. It turns out that under appropriate circumstances, this property for Y is inherited from $(X_I)_\kappa$ itself. The key to the argument is an appropriate fragment of the vast theory of the partition calculus, the creation of Erdős and Rado. The required definitions and theorems are laid out in detail in [7] and [19]; see also [11] for an extended treatment. For our purposes, we need only the notational conventions given in Notations 3.5 and 3.6, together with Theorem 3.7.

Notation 3.5. For $\alpha \geqslant \beta$, the notation $\alpha \to \Delta(\kappa, \beta)$ means that for every family of sets $\{S_\xi\colon \xi < \alpha\}$ indexed (not necessarily faithfully) by α, with each $|S_\xi| < \kappa$, there are $A \in [\alpha]^\beta$ and a (possibly empty) set J such that $S_\xi \cap S_{\xi'} = J$ whenever $\{\xi, \xi'\} \in [A]^2$.

Many authors refer to a family $\{S_\xi\colon \xi \in A\}$ as in Notation 3.5 as a *quasi-disjoint* family, or a *Δ-system*, with *root* J.

Notation 3.6. The notation $\kappa \ll \alpha$ means (a) $\kappa < \alpha$ and (b) if $\lambda < \kappa$ and $\beta < \alpha$, then $\beta^\lambda < \alpha$.

The relation $(2^\alpha)^\alpha = 2^\alpha$ reflects to the fact that $\alpha^+ \ll (2^\alpha)^+$. That example motivated the introduction of the more general notation. The basic combinatorial theorem, given in [12] for successor cardinals α and in [13] in general, is this.

Theorem 3.7. *If $\kappa \ll \alpha$ and α is regular, then $\alpha \to \Delta(\kappa, \alpha)$.*

For a proof of Theorem 3.7, see [12] and [13], or [7, 1.4].

The following result, described in [7, 3.6(c) and 3.8(c)], allows the transfer of properties of pseudocompactness type from spaces of the form $(X_J)_\kappa$ to certain dense subspaces of $(X_I)_\kappa$. Even the case $Y = X_I$ is worthy of note.

Theorem 3.8. *Let $\alpha \to \Delta(\kappa,\alpha)$ with $\alpha \geqslant \kappa$ and let $Y \subseteq (X_I)_\kappa$. If $(X_J)_\kappa$ is pseudo-(α,κ)-compact for all $\emptyset \neq J \in [I]^{<\kappa}$, and if also $\pi_J[Y] = X_J$ for all $\emptyset \neq J \in [I]^{<\kappa}$, then Y is pseudo-(α,κ)-compact.*

Proof. Given a family $\{U_\xi\colon \xi < \alpha\}$ of basic open subsets of $(X_I)_\kappa$, one finds $p \in Y$ such that each basic open neighborhood U of p in $(X_I)_\kappa$ satisfies $U \cap U_\xi \neq \emptyset$ for at least κ-many $\xi < \alpha$. This suffices, since (because Y is dense in $(X_I)_\kappa$) from $U \cap U_\xi \neq \emptyset$ follows $U \cap U_\xi \cap Y \neq \emptyset$. Since $|R(U_\xi)| < \kappa$ for each $\xi < \alpha$, there are $A \in [\alpha]^\alpha$ and a set J such that $R(U_\xi) \cap R(U_{\xi'}) = J$ whenever $\{\xi,\xi'\} \in [A]^2$. If $J = \emptyset$, any $p \in Y$ is as required.

For $J \neq \emptyset$, we consider the family of open subsets $\{\pi_J[U_\xi]\colon \xi \in A\}$ of $(X_J)_\kappa$. Since $(X_J)_\kappa$ is pseudo-(α,κ)-compact, there is a point $p' \in X_J$ such that every neighborhood V in $(X_J)_\kappa$ of p' satisfies $V \cap \pi_J[U_\xi] \neq \emptyset$ for at least κ-many $\xi \in A$. Choose any $p \in Y$ with $p_J = p'$, and for U a basic open neighborhood of p in $(X_I)_\kappa$ define $A' := \{\xi \in A\colon \pi_J[U] \cap \pi_J[U_\xi] \neq \emptyset\}$. Then, write $A' = B' \cup C'$ with $B' := \{\xi \in A'\colon U \cap U_\xi = \emptyset\}$ and $C' := \{\xi \in A'\colon U \cap U_\xi \neq \emptyset\}$. If $\xi \in B'$ then $(R(U) \cap R(U_\xi)) \backslash J \neq \emptyset$ (as in [7, 3.1]), so there is $i_\xi \in (R(U) \cap R(U_\xi)) \backslash J$. The map $\xi \to i_\xi$ from B' into $R(U)$ is injective, so from $|A'| \geqslant \kappa$ and $|B'| < \kappa$ it follows that $|C'| \geqslant \kappa$, as required. □

Corollary 3.9. *Let $\kappa \ll \alpha$ with α regular and let $Y \subseteq X_I$ be such that $\pi_J[Y] = X_J$ whenever $\emptyset \neq J \in [I]^{<\alpha}$. If $(X_J)_\kappa$ is pseudo-(α,κ)-compact for every $\emptyset \neq J \in [I]^{<\kappa}$, then Y is* **M***-embedded in $(X_I)_\kappa$.*

Proof. We have $\alpha \to \Delta(\kappa,\alpha)$ from Theorem 3.7, so in the topology inherited from $(X_I)_\kappa$ the space Y (by Theorem 3.8) is pseudo-(α,κ)-compact. Then according to Lemma 2.1 each continuous $f : Y \to Z$ with Z metrizable depends on $< \alpha$-many coordinates, so Theorem 3.2 applies. □

Specializing to the case $\kappa = \omega$ and $\alpha = \omega^+$, we have a result which uses a more familiar vocabulary. (The proof that every Lindelöf space is pseudo-ω^+-compact is routine; see [7, 10.6(b)] for a stronger result.)

Corollary 3.10. *If $Y \subseteq X_I$ and $\pi_J[Y] = X_J$ for all $\emptyset \neq J \in [I]^{<\omega^+}$, and if each X_J, for $\emptyset \neq J \in [I]^{<\omega}$, is Lindelöf, then Y is* **M***-embedded in X_I.*

Corollaries 3.9 and 3.10 may be compared with Theorem 3.11, a result of Engelking [9]: After strengthening the hypothesis on Z, we achieved a stronger conclusion, even in the case $\kappa = \omega$. We note again that in contrast with Theorem 3.11, no separation axioms for the spaces X_i $(i \in I)$ are imposed in Corollaries 3.9 and 3.10.

Theorem 3.11. *(See [9, Theorem 1].) If X_I is a product of T_1-spaces such that each X_J, for $\emptyset \neq J \in [I]^{<\omega}$, is Lindelöf, Z is a Hausdorff space with G_δ-diagonal, and $p \in X_I$, then each $f \in C(\Sigma(p), Z)$ depends on countably many coordinates and extends to $\overline{f} \in C(X_I, Z)$.*

Theorem 3.12. *Let $\kappa \ll \alpha$ with α regular, let $\{X_i\colon i \in I\}$ be a set of topologically complete spaces, and let $Y \subseteq X_I$ be such that $\pi_J[Y] = X_J$ whenever $\emptyset \neq J \in [I]^{<\alpha}$. If $(X_J)_\kappa$ is pseudo-(α,κ)-compact for every $\emptyset \neq J \in [I]^{<\kappa}$, then $(X_I)_\kappa = \gamma(Y)$.*

Proof. It is enough to know that Y is **M**-embedded in $(X_I)_\kappa$ and that $(X_I)_\kappa$ is topologically complete. The first requirement is given by Corollary 3.9, while the second is a theorem given by Kato [20, 4.2(2)] and later, with a different proof, by S. Williams [31, 2.5(2)]: When given a κ-box topology, a product of topologically complete spaces remains topologically complete. □

The following definitions are consistent with the conventions of [6] and [7]; see also [19] and [10] for alternative terminology.

Definition 3.13. Let $X = (X, \mathcal{T})$ be a space.

(a) $d(X)$, the *density character of* X is $\min\{|D|\colon D$ is dense in $X\}$.
(b) $S(X)$, the *Souslin number of* X, is $\min\{\alpha\colon$ no $\mathcal{U} \in [\mathcal{T}]^\alpha$ is pairwise disjoint$\}$.

(c) X is *α-compact* if for every cover $\mathcal{U} \subseteq \mathcal{T}$ of X there is a subcover $\mathcal{V} \in [\mathcal{U}]^{<\alpha}$.
(d) X is *weakly α-compact* if for every cover $\mathcal{U} \subseteq \mathcal{T}$ of X there is a subfamily $\mathcal{V} \in [\mathcal{U}]^{<\alpha}$ such that $\bigcup \mathcal{V}$ is dense in X.

The following result closely parallels Corollary 10.7(a) of [7], and it improves that statement in two ways: we deal here with topologically complete spaces rather than with realcompact spaces, and the cardinal number $\alpha = \omega$ of [7, 10.7(a)] is replaced by arbitrary $\alpha \geqslant \omega$.

Theorem 3.14. *Let α be an infinite cardinal, $\{X_i: i \in I\}$ be a set of topologically complete spaces, and $Y \subseteq X_I$ satisfy $\pi_J[Y] = X_J$ whenever $\emptyset \neq J \in [I]^{\leqslant\alpha}$. Suppose that X_J, for each $\emptyset \neq J \in [I]^{<\omega}$, satisfies either*

(i) $d(X_J) \leqslant \alpha$; *or*
(ii) $S(X_J) \leqslant \alpha^+$; *or*
(iii) X_J *is* α^+*-compact*; *or*
(iv) X_J *is weakly* α^+*-compact*; *or*
(v) X_J *is pseudo-*α^+*-compact space.*

Then $X_I = \gamma(Y)$.

Proof. One sees easily, as in [7] (the case $\alpha = \omega$), that properties (i)–(v) are related by the implications (i) ⇒ (ii) ⇒ (iv) ⇒ (v) and (iii) ⇒ (iv). Since α^+ is regular and satisfies $\omega \ll \alpha^+$, the desired conclusion is immediate from Theorem 3.12 (upon replacing α there by α^+). □

Remarks 3.15. (a) It follows from the Katětov–Shirota Theorem (cf. [14, 15.20], [6, 6.3]) that every topologically complete space of non-Ulam-measurable cardinal is realcompact. Thus if in Theorems 3.12 and 3.14 each X_i is of non-Ulam-measurable cardinal—in particular, if no uncountable measurable cardinals exist—then those theorems become simply statements about realcompact spaces (with the conclusion $X_I = \upsilon(Y)$). The case $\alpha = \omega$ of Theorem 3.14 is given in ZFC in [7, 10.7]. In contrast, if an Ulam-measurable cardinal α exists and each X_i $(i \in I)$ is the discrete space with $|X_i| = \alpha$, then $X_i = \gamma(X_i) \neq \upsilon(X_i)$ and [7, 10.7] provides no information. Indeed, taking for simplicity $Y = X_I$ we have $X_I = \gamma(Y)$, while $X_I = \upsilon(Y)$ is false.

(b) From personal experience, the reader will be acquainted with questions of this form, given a class $\mathbb{P}$ of spaces: (1) If $X_0, X_1 \in \mathbb{P}$, must $X_0 \times X_1 \in \mathbb{P}$? (2) If $\{X_i: i \in I\} \subseteq \mathbb{P}$ and each $X_J \in \mathbb{P}$ when $\emptyset \neq J \in [I]^{<\omega}$, must $X_I \in \mathbb{P}$? We do not attempt here to recapitulate the vast literature relating to these questions as they pertain to properties (i)–(v) in Theorem 3.14, but (restricting attention for simplicity to the case $\alpha = \omega$) we do remark: For (i), (1) holds and (2) fails (both these statements are obvious); for (ii), (1) is independent of the axioms of ZFC [18], [5, 3.15], [7, pp. 201–204] and (2) holds [5, 3.11], [7, 3.6(a)]; for (iii), (1) fails (use the Sorgenfrey line) and (2) fails [26]; for (iv), (1) fails [16] and (2) holds [29]; for (v), (1) fails [24] and (2) holds [7, 3.6(c)].

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