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Pseudocompact groups: progress and problems ☆

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Abstract

Several months ago the speaker and Jan van Mill gave a proof of this result [W.W. Comfort, J. van Mill, Extremal pseudocompact abelian groups are compact metrizable, Abstracts Amer. Math. Soc. 27 (2006) 78 (Abstract #1014-22-958); W.W. Comfort, J. van Mill, Extremal pseudocompact abelian groups are compact metrizable, Proc. Amer. Math. Soc. 135 (2007) 4039–4044]: A pseudocompact abelian group of uncountable weight admits both a proper dense pseudocompact subgroup and a strictly larger pseudocompact group topology.

This presentation will describe both the necessary new details of the argument and the historical development (useful tools, special cases). Among those who contributed essentially are: K.A. Ross (1964, 1966); T. Soundararajan (1982); L.C. Robertson (1982, 1988); J. van Mill (1989); J. van Mill and H. Gladdines (1994); J. Galindo (2002).

Several related unsolved problems will be cited.
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1. Introduction (historical remarks)

I appreciate the invitation from the Organizing Committee to speak here today. I take it as a particular privilege to be invited to open the scientific program of the 10th Prague Topological Symposium. Before turning to mathematics proper, I want to devote three minutes of my allocated time to historical commentary. This will be unnecessary for those of you with a social, global awareness, back in 1961. But I estimate that over half of us in this room today were not even alive when the first Toposym Prague was convened. A brief history lesson may be appropriate.

To be brief: the world in the 1960's and the 70's and the 80's was more constrained and constricted than it is today. With no EU, with NATO oriented more militaristically than economically, with no e-mail, with regular mail between countries subject to governmental inspection and to unannounced delays of unpredictable duration, there were few reliable avenues for social and scientific exchange among average citizens not associated with the diplomatic corps. In this difficult climate, constricted by the harshest regulations imaginable with respect to international travel, leading Czechoslovak topologists under the initiative of Eduard Čech performed a service for the international topological community whose importance cannot be overstated. There was much joy as topologists from Czechoslovakia, the USSR, Poland, Hungary, Bulgaria, East Germany, Romania and Yugoslavia met with each other face to face for

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the first time, and with their Western colleagues. As the torch was passed from such visionaries as Academicians Josef Novák and Miroslav Katětov, also Vlastimil Pták, to younger hands like Zdeněk Frolík and Miroslav Hušek, political conditions remained constrained even while improving. Though the sense of urgency has given way now to a more relaxed social and scientific climate, I wanted to remind you of the absolutely essential service rendered to our profession by this sequence of meetings. If topology as we know it is healthy and vital today, that is in substantial measure because of the energy, the vision, indeed the courage, of Czechoslovak mathematicians who years ago opened doors and facilitated passage for us today.

2. Introduction (mathematical)

Now for some mathematics.

The term *pseudocompact* was introduced in 1948 by Edwin Hewitt [34] to refer to a topological space on which each real-valued continuous function is bounded. It turns out [19] (1.1) that if a topological group G is pseudocompact, then it is *totally bounded*, or *precompact*, if you prefer, which for us today means just that it embeds as a dense subgroup of a compact topological group [39]. This latter, the *Weil completion* of G, is unique in the obvious sense; we denote it here by the symbol $\overline{G}$. Is every dense subgroup of a compact group necessarily pseudocompact in Hewitt's sense? Certainly not, look, for example, at the torsion subgroup of the circle, which is homeomorphic as a space to the rational numbers; on the rationals, there are many unbounded real-valued continuous functions, for example the identity function. Ken Ross and I in 1966 showed this:

(1) *(See [19].) A dense subgroup of a compact group is pseudocompact if and only if it is G_δ-dense* (*in the sense that it meets every nonempty G_δ subset*).

This then follows easily:

(2) *(See [19].) A dense subgroup of a pseudocompact group is itself pseudocompact if and only if it is G_δ-dense.*

Now I will list two questions which arise immediately and which have claimed a good bit of attention in the literature over the years. For ease of reference later I label these Questions I and II, but I emphasize that they are in fact no longer Questions; they have been solved. This lecture is devoted to the history and the nature of their solutions.

Question I. Does every pseudocompact topological group have a proper G_δ-dense subgroup—i.e., a proper dense pseudocompact subgroup?

The companion question has a very different flavor. But it turns out to be unexpectedly susceptible to investigation using similar tools:

Question II. If a group admits a pseudocompact topological group topology, does it admit a strictly larger one?

Near the end of this talk I will touch briefly on what little is known about these questions for nonabelian groups. My principal goal today is to explain to you that, once the obvious negative case is disposed of, the answer to both those questions for abelian groups is "Yes".

The obvious negative case is that of a metrizable group, equivalently, of a group with countable weight. Such a group, if pseudocompact, is easily seen to be compact metric [34] (Theorem 30), [30] (Exercise 3D2), [20] (3.1), and therefore it has no proper dense pseudocompact subgroup and it admits no properly larger pseudocompact group topology [38] (4.6), [20] (3.1), [15] (3.1), [17] (2.4). So the theorem for discussion today is this.

Theorem. *(See [9,10].) A nonmetrizable pseudocompact abelian group has both a proper dense pseudocompact subgroup and a properly larger pseudocompact topological group topology.*

I have been working with smart, creative, co-authors on approaches to this theorem ever since it first began to take shape over 20 years ago [1] (Questions 4D, 5C), [2] (3.1), [20,15,17]. As I indicated, the investigation reached its final successful conclusion less than a year ago in joint work with Jan van Mill.

I am going to mention now six co-authors and, in each case, one or two facts we uncovered, or tools we developed, which proved useful in the final solution. With this background, I hope that at least the main ideas of the proof of the theorem will become transparent.

I have already mentioned Ken Ross and the paper [19] we wrote in 1966. Recall that every pseudocompact topological group is totally bounded. In earlier work in 1964, Ross and I had shown:

(3) *(See [18].)* (a) *On an abelian group G, every totally bounded group topology $\mathcal{T}$ is the topology induced by some point-separating group A of homomorphisms from G to the circle.* [We write $\mathcal{T} = \mathcal{T}_A$.] *Furthermore,* $\widehat{(G, \mathcal{T}_A)} = A$. Consequently
(b) *given* $(G, \mathcal{T})$ *as in* (a), *only one such A satisfies* $\mathcal{T} = \mathcal{T}_A$.

The result (3) brings to the foreground two attractive unsolved questions. They are peripheral to today's considerations, however, so I will take time only to list for you those relevant citations from the literature of which I am aware, but not to summarize the known results. It is enough for now to state that neither has been solved in full generality. [**Note.** Neither of these questions is new. The articles [3] (3.5, 5.9), [4] (Problems 2.2, 3.2), and [27] (Problem 9) provide some references and discussion.]

Question 1. Which (abelian) groups admit a pseudocompact topological group topology?

[Some relevant citations: [28,24,25,11,12,21,26].]

Question 2. Let G be an abelian group which does admit a pseudocompact topological group topology. Which point-separating groups of homomorphisms from G to the circle induce such a topology on G?

[Some relevant citations: [7] (4.1), [17] (5.11, 6.5), [29], [33] (§§3, 4), [5] (3.6, 3.10), [36].]

One fact related to these questions is clear from (3) above, and it is crucial to the arguments to follow: If a pseudocompact abelian topological group $G = (G, \mathcal{T}) = (G, \mathcal{T}_A)$ is to have a strictly larger pseudocompact group topology, that will take the form $\mathcal{T}_B$, with B a group of homomorphisms properly containing A. So one may ask: Given $G = (G, \mathcal{T}) = (G, \mathcal{T}_A)$, when does $\mathrm{Hom}(G, \mathbb{T})$ contain another homomorphism h, not $\mathcal{T}_A$-continuous, which may be adjoined to A so that the new topology $\mathcal{T}_{\langle A \cup \{h\}\rangle}$ is pseudocompact? Does every $\mathcal{T}_A$ admit such a (discontinuous) h?

Notice that in general, even if G is one of those abelian groups which does admit a pseudocompact group topology, there may be some homomorphisms $h \in \mathrm{Hom}(G, \mathbb{T})$ which cannot belong to any B for which $(G, \mathcal{T}_B)$ is pseudocompact. After all, if $h \in B$ then h is $\mathcal{T}_B$-continuous, and the continuous image of a pseudocompact space is pseudocompact, so if $(G, \mathcal{T}_B)$ is pseudocompact then $h[G]$ must be a pseudocompact subgroup of the circle, i.e., a compact subgroup of the circle, i.e., either $\mathbb{T}$ itself or one of its finite subgroups. In particular: if $h[G]$ is a proper, infinite subgroup of $\mathbb{T}$, for example, if $|h[G]| = \omega$, then there is no pseudocompact group topology $\mathcal{T}_B$ on G such that $h \in B$.

We see that looking to expand or refine $\mathcal{T} = \mathcal{T}_A$ to a larger pseudocompact group topology on G is equivalent to finding a discontinuous homomorphism, i.e., a homomorphism not in A, which can be adjoined so that $(G, \mathcal{T}_B)$ remains pseudocompact with $B = \langle A \cup \{h\}\rangle$.

My next hero in this story is the Indian mathematician T. Soundararajan, who visited Wesleyan University in 1981 under the auspices of AID, the Agency for International Development. In our work on some related questions [20], we came to recognize the utility, when investigating (pseudocompact) topological groups G, of the following class of subgroups of G:

Notation. $\Lambda(G) := \{N\colon\ N$ is a closed, normal G_δ-subgroup of $G\}$.

Soundararajan and I developed this characterization of pseudocompactness.

(4) *(See [20] (3.3).) For a topological group G, these conditions are equivalent.*
(a) *G is pseudocompact*;

(b) *$N \in \Lambda(G) \Rightarrow G/N$ is compact*;
(c) *$N \in \Lambda(G) \Rightarrow G/N$ is compact metrizable.*

Lew Robertson and I noticed later that if G satisfies the conditions of (4) then so does each $N \in \Lambda(G)$. The proof is routine: Suppose that G is pseudocompact, let $N \in \Lambda(G)$, and let $M \in \Lambda(N)$. Then $M \in \Lambda(G)$, so G/M is compact metric, so N/M is compact metric. So N is pseudocompact.

Restated:

(5) *(See [17] (6.2).) If G is pseudocompact and $N \in \Lambda(G)$, then N is pseudocompact.*

So far as I am aware, the first explicit results in the literature about topological groups with a proper dense pseudocompact subgroup and with a strictly larger pseudocompact group topology are in [20] (Theorems 4.3 and 4.4), where it is shown that nonmetrizable, totally disconnected compact abelian groups have both properties. Robertson and I extended that result with this theorem. I want to say a few words about the proof because, as we will note later, the argument which gives (6)(c) is available, *mutatis mutandis*, in a broader context.

(6) *(See [15].) Let $G = (G, \mathcal{T})$ be a compact abelian topological group, with $w(G) > \omega$. Then*
(a) *G maps by a continuous homomorphism ϕ onto a group of the form $K = S^\kappa$ with $\kappa > \omega$, S a compact subgroup of $\mathbb{T}$*;
(b) *there is $h \in \mathrm{Hom}(G, \mathbb{T})$ such that $\mathrm{graph}(h)$ is G_δ-dense in $G \times S$*; *and*
(c) *G admits both a proper, dense, pseudocompact subgroup and a stronger pseudocompact group topology.*

Proof. (Outline) (a) The dual group $\widehat{G}$ of G satisfies $\omega < wG = |\widehat{G}| = r(\widehat{G})$ [35] (24.15(i)), so either $r_0(\widehat{G}) > \omega$ or there is a prime p such that $r_p(\widehat{G}) > \omega$. (a) then follows using Pontrjagin duality, as in [35] (§24), [15] (3.2(b)) or [13] (5.4).

(b) and (c) Given G, K, ϕ and S as in (a), define $\psi : \bigoplus_\kappa S \to S \subseteq \mathbb{T}$ by $\psi(s) = \prod_{\eta<\kappa} s_\eta$ (a finite product) and extend ψ to a homomorphism (also denoted ψ) from K to $\mathbb{T}$. Then $h := \psi \circ \phi \in \mathrm{Hom}(G, \mathbb{T})$, and it is easy to see (using $\kappa > \omega$) that $\mathrm{graph}(h)$ is G_δ-dense in $(G, \mathcal{T}) \times S$, hence by (3) is pseudocompact. (In particular, $\ker(h)$ is G_δ-dense in $(G, \mathcal{T})$.) The map $i : G \twoheadrightarrow \mathrm{graph}(h)$ given by $i(x) = \langle x, h(x)\rangle$ is an isomorphism, and the topology $\mathcal{U}$ on G defined by the requirement that $i : (G, \mathcal{U}) \twoheadrightarrow \mathrm{graph}(h) \subseteq (G, \mathcal{T}) \times S$ is a homeomorphism is a pseudocompact group topology on G properly refining $\mathcal{T}$. $\square$

It can be noted that once (a) of (6) is known, the existence of a proper dense pseudocompact subgroup H of $(G, \mathcal{T})$ may be seen directly: the group K has such a subgroup D—for example, one may take for D the usual Σ-product inside K—and then $H := \phi^{-1}(D)$ fills the bill. (The group H "inherits" from D the property that each of its countable subsets has compact closure, a property stronger than pseudocompactness, and the surjection $\phi : G \twoheadrightarrow K$ is an open map [35] (5.29).)

According to (6)(c), then, Questions I and II had been solved completely in 1982, in the case that the given infinite abelian group G is compact.

Let me recapitulate and clarify just where we stand, and fix some notation and terminology to be used in the rest of this talk. $G = (G, \mathcal{T})$ is an infinite, abelian group, pseudocompact and nonmetrizable. Taking a queue from [17], we say as in [3,5] that $(G, \mathcal{T})$ is *refinement extremal* if no pseudocompact group topology on G properly contains $\mathcal{T}$, and G is *subgroup extremal* if G admits no proper, dense, pseudocompact subgroup. In that language, the theorem being discussed today is that no G is extremal in either of those two senses.

You see already at least one bit of evidence substantiating the assertion that Questions I and II are related: When $G = (G, \mathcal{T}) = (G, \mathcal{T}_A)$ is compact, some $h \in \mathrm{Hom}(G, \mathbb{T})$ with $h \notin A$ can indeed be adjoined to become continuous in a strictly larger pseudocompact group topology; and, h may be chosen so that $\ker(h)$ is (proper and) G_δ-dense in G.

In 1988, Lew Robertson and I solved Questions I and II for torsion groups G [17] (7.5). Then in 1989 Jan van Mill and I [8] solved Question I for many connected pseudocompact topological groups, and in 1992 (published in 1994) with his graduate student Helma Gladdines [6] we solved it completely for groups G which are big in the sense that $|G| > \mathfrak{c}$, or which are small in the sense that $w(G) \leqslant \mathfrak{c}$.

After these developments, I spent some years working alone on this project, and without any noteworthy success. That changed when Jorge Galindo visited Wesleyan in 1999. Among the new facts to emerge under his guidance were these:

(7) *(See [29] (7.3), [5] (5.10(c)).) If $r_0(G) > \mathfrak{c}$, then G is neither subgroup extremal nor refinement extremal.*
(8) *(See [29] (6.4(2)), [5] (5.7).) If G is either subgroup extremal or refinement extremal, then there is a large connected subgroup of G; indeed, for every $N \in \Lambda(G)$ there is a connected $M \in \Lambda(N) \subseteq \Lambda(G)$.*

Item (8) really knocked me out. We are, after all, considering arbitrary pseudocompact abelian groups (of course, nonmetrizable, and extremal in some sense). How in the world, and why, does connectivity arise in this context? With respect to this result, I went through stages of thinking which probably most of us have experienced at one time or another in some mathematical context. Stage 1: That is a shock and a surprise. Stage 2: That cannot be true, the proof must be wrong, I will have a counterexample tomorrow. Stage 3, approximately one week later: Of course it is true, it is obvious.

Somehow I still do not believe (8), but let me show you why it is obvious. The key is Mycielski's observation [37], a consequence of Pontrjagin duality: A compact abelian group is connected if and only if it is divisible. I believe we have time to outline a proof of (8).

Proof. We have $N \in \Lambda(G)$, so $\overline{N} \in \Lambda(\overline{G})$ [8] (2.7(c)), so $n\overline{N} \in \Lambda(n\overline{G})$. Then, take $H := \bigcap_n n\overline{N} \in \Lambda(\overline{G})$. This H is compact and divisible, hence connected, and then $M := H \cap G \in \Lambda(G)$ is connected since $H = \beta(M)$ [19] (1.2, 4.1). From $M \in \Lambda(G)$ and $M \subseteq N$ follows $M \in \Lambda(N)$. $\square$

It follows easily from (8), as Galindo and I noted [29,5], that no pseudocompact abelian group can be "so subgroup extremal" that all $N \in \Lambda(G)$ can be subgroup extremal. In other words, we could see in 1999 that even though there might be some G with no proper, dense pseudocompact subgroup (we did not know, at that time), even for such G there is some $N \in \Lambda(G)$ which does have a proper, dense, pseudocompact subgroup. I thought that with this result we should be home free, at least with respect to Question I. Let me show you the naïve line of approach which looks promising but does not work; doing this will not waste your time—it provides a helpful introduction and transition to what does work.

Take G, then $N \in \Lambda(G)$ which is not itself subgroup extremal, then take a proper, G_δ-dense subgroup D of N. Let X select one point from each coset of N in G. Then $H := \langle D \cup X \rangle$ is a G_δ-dense subgroup of G.

Does that simple argument solve Question I? No, the constructed group H is pseudocompact alright, but it may fail to be proper.

What does work is this—and this is one of the ingredients uncovered last year with Jan van Mill. (One should note that this result, essential to our approach, was found also by Dikranjan, Giordano Bruno, and Milan [23] (4.11).) Suppose not only that some $N \in \Lambda(G)$ has a proper, dense pseudocompact subgroup D, but also that D may be chosen so that $r_0(N/D) \geqslant \mathfrak{c}$. Then, there is a selection set X for the coset space G/N, constructed recursively with some care, such that the pseudocompact subgroup $H := \langle D \cup X \rangle$ of G is proper, indeed it is proper for the good reason that $r_0(G/H) \geqslant \mathfrak{c}$. This fulfills two purposes at once: It obviously gives a proper, dense, pseudocompact subgroup of G, thus responding to Question I. But also, settling Question II, since $\mathbb{T}$ is divisible there is a surjective homomorphism $\phi : G/H \twoheadrightarrow \mathbb{T}$, and then $h := \psi \circ \phi_H : G \twoheadrightarrow \mathbb{T}$ is a homomorphism, not continuous on G, which (much as in the proof above of (6)(c)) can be adjoined while retaining pseudocompactness.

So where do we stand? According to [17] (7.5) and (7), respectively, no G with $r_0(G) = 0$ or with $r_0(G) > \mathfrak{c}$ is extremal in either sense. It is well known and easily seen for an arbitrary pseudocompact abelian group G that if $r_0(G) > 0$ then $r_0(G) \geqslant \mathfrak{c}$ ([8] (2.17), [25] (2.17 and 3.17), [26] (3.8), [5] (4.3)), so it becomes, finally, sufficient to prove this statement.

(9) *Let G be a nonmetrizable, pseudocompact abelian group such that $r_0(G) = \mathfrak{c}$. If G is either subgroup extremal or refinement extremal, then there is $N \in \Lambda(G)$ with a proper, dense pseudocompact subgroup D such that $r_0(N/D) = \mathfrak{c}$.*

Proof. Use (8) to choose a connected group $C \in \Lambda(G)$. Then $\overline{C}$ is compact and connected, hence divisible, and we write $\overline{C} = \bigoplus_\kappa \mathbb{Q} \oplus t\overline{C}$, with t denoting the torsion subgroup. Say $C \subseteq \bigoplus_{s\in S} \mathbb{Q}_s + t\overline{C}$ with S minimal, $|S| = \mathfrak{c}$. For $A \subseteq S$ set $G(A) := C \cap (\bigoplus_{s\in A} \mathbb{Q}_s \oplus t(\overline{C}))$, and define $\mathcal{A} := \{A \subseteq S\colon \exists N \in \Lambda(G), N \subseteq G(A)\}$. Then $\mathcal{A}$ is a filter on S, closed under countable intersections, so $\mathcal{A}$ is not an ultrafilter on S since $\mathfrak{c}$ is not an Ulam-measurable cardinal, so we can partition $S = \bigcup_n S_n$ with each $|S_n| = \mathfrak{c}$, indeed such that $A \in \mathcal{A}, n < \omega \Rightarrow |A \cap S_n| = \mathfrak{c}$. Let $V_n := G(\bigcup_{m\leqslant n} S_m)$, so $V_n \subseteq V_{n+1}$ and $C = \bigcup_n V_n$. Now C in its G_δ topology satisfies the conclusion of the Baire category theorem [16] (2.4), so there are $N \in \Lambda(C)$ and $n < \omega$ such that $D := V_n \cap N$ is G_δ-dense in N. Then, $r_0(N/D) = \mathfrak{c}$. □

That concludes the proof of the theorem that every pseudocompact abelian group of uncountable weight admits both a proper, dense, pseudocompact subgroup and a strictly larger pseudocompact group topology.

3. Suggestions for further work

The result is definitive, but as usual in mathematics it gives rise to additional questions. I will close by mentioning four of these. Since already in Section 3 I mentioned two unsolved problems worthy of attention, I give these the numbers 3 through 6.

Question 3. Is the parallel statement valid, when G is not assumed abelian?

As to r-extremality, the answer is affirmative when there are a pseudocompact group F with $|F| > 1$ and a cardinal $\kappa > \omega$ such that $G = F^\kappa$ [13] (3.4) or when G is compact and connected [13] (6.7); more refined results about the poset of pseudocompact refinements of a compact, connected group $(G, \mathcal{T})$ with $\mathrm{cf}(w(G, \mathcal{T})) > \omega$ are given in [14] (6.7, 7.2). In any event there are obstructions to approaching Question 3 by simply mimicking the proof given in the abelian case. Pontrjagin duality appears on stage frequently in the treatment I have outlined here, though in the background, and that of course is no longer available. But powerful tools have been developed which can stand in its place in different nonabelian mathematical settings. And happily one of the present tools, the theorem [19] that a dense subgroup of a compact group is pseudocompact if and only if it is G_δ-dense, remains fully valid even without the abelian hypothesis. In any event, we have proved two statements here about pseudocompact abelian groups, one concerning subgroups and one concerning refinements, and I believe that no counterexample is known to either of those when the abelian hypothesis is omitted.

In some earlier works with some of the co-authors I have cited, before the general result had been established, we asked whether the two sorts of extremal behavior under consideration were equivalent for pseudocompact abelian groups. In our present state of ignorance about the nonabelian case, that question may reasonably be resuscitated.

Question 4. Let G be a (possibly nonabelian) pseudocompact topological group. Are the conditions

(a) *G is subgroup-extremal,*
(b) *G is refinement-extremal*

equivalent?

As indicated above, a metrizable group (whether or not abelian) is pseudocompact if and only if it is compact. Since such groups are extremal in both senses, it is natural to restrict Question 4 to groups of uncountable weight.

Before the results of [10] became available, researchers in Udine, Italy, had identified and studied weak versions of metrizability in the context of extremality, thus obtaining in a uniform and symmetric manner most of the previously known results about r- and s-extremal pseudocompact abelian groups. Concatenating definitions and results from [31,23,22,32], let us say that a pseudocompact abelian group G is *c-extremal* [resp., *singular*] if every dense, pseudocompact subgroup H of G has $r_0(G/H) < \mathfrak{c}$ [resp., if there is an integer $m > 0$ such that mG is metrizable]. The utility of these ideas in the study of extremal pseudocompact groups is evident from this theorem, shown in [31,23]: If some $N \in \Lambda(G)$ is not c-extremal then G itself is not c-extremal (hence, is neither r- nor s-extremal). This prompted Giordano Bruno to raise these questions (see also [22] (§1) for additional motivational discussion).

Question 5. (See [32].) Which pseudocompact abelian groups G admit a proper, dense pseudocompact subgroup H such that

(a) *H is totally dense in G?*
(b) *H is essential in G?*

[As usual, H here is *totally dense* in G if it meets every closed (normal) subgroup of G in a relatively dense subset; and H is *essential* in G if it meets nontrivially every nontrivial closed (normal) subgroup of G.]

Parts (a) and (b) of Question 5 have been solved completely when G is compact, in [22] and [32], respectively.

Question 6. For arbitrary topological classes $\mathbb{P}$ and $\mathbb{Q}$, one may ask whether every topological group $G \in \mathbb{P}$ admits a dense subgroup and/or a strictly larger group topology, in $\mathbb{Q}$. (Parts of [3] and [4] are framed in this context.) Here, we have treated the case $\mathbb{P} = \mathbb{Q} =$ pseudocompact abelian. It is reasonable to inquire into what our methods say, if anything, about other classes. In conversation, Jan van Mill has offered some hope in this direction. I am a pessimist, on the grounds that the G_δ-density property (2) of pseudocompact topological groups permeates every aspect of our proofs. That property precisely characterizes pseudocompactness, nothing more and nothing less. But Jan points out in rejoinder that to some extent our argument separates the algebra from the topology, and the algebra may be strong enough to carry the day for some other classes of topological groups.

In the past, when arguing with my co-authors on some mathematical point, I have found that the odds that they are right, and I am wrong, are about 7 to 2 in their favor; in the case of Jan van Mill, this may rise to more like 25 to 1. I hope that once again he is right. Time will tell.

I thank you for your attention.

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