EXTREMAL PSEUDOCOMPACT ABELIAN GROUPS
ARE COMPACT METRIZABLE

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Abstract. Every pseudocompact Abelian group of uncountable weight has both a proper dense pseudocompact subgroup and a strictly finer pseudocompact group topology.

1. Introduction

All topological groups here are assumed to satisfy the Hausdorff separation axiom. A pseudocompact group $G$ is said to be $r$-extremal [resp. $s$-extremal] if $G$ admits no strictly finer pseudocompact group topology [resp. $G$ has no proper dense pseudocompact subgroup]. Early formulations of these notions appeared in [6], [7]. From the fact that a pseudocompact space of countable weight is compact and metrizable it follows readily (as in [7, 2.3]) that every pseudocompact group of countable weight is both $r$-extremal and $s$-extremal. It is natural to ask whether there are extremal pseudocompact groups of uncountable weight. This question has generated much attention during the last two decades. See [1], [12] and [10] for more information. An affirmative answer was given in [7] for zero-dimensional Abelian groups. In [2] it was shown that no pseudocompact Abelian group of cardinality greater than $c$ is $s$-extremal. For partial answers in the class of connected groups, see for example [3], [12] and [1].

The aim of this paper is to answer the question for Abelian groups.

Theorem 1.1. A pseudocompact Abelian group of uncountable weight is neither $r$-extremal nor $s$-extremal.

We keep this presentation short by invoking several essential results established in the literature. We plan in [5] to present a polished, complete and self-contained proof of Theorem 1.1.

We announced our results at the annual meeting of the American Mathematical Society in January, 2006 [4].

2. Preliminaries

In this section we fix notation and we cite the results we need from the literature.
The symbol \( wX \) denotes the weight of a topological space \( X \). A subspace of a space \( X \) is \( G_\delta \)-dense in \( X \) if it meets every nonempty \( G_\delta \)-subset of \( X \). If \( X \) is a set and \( \kappa \) a cardinal number, then \( |X|^\leq \kappa \) denotes \( \{A \subseteq X : |A| \leq \kappa \} \).

For Abelian groups we use additive notation. Let \( G \) be an Abelian group. If \( A \subseteq G \), then \( \langle A \rangle \) denotes the subgroup of \( G \) generated by \( A \). A subset \( X \) of \( G \) is called independent if for every \( x \in X \) we have \( \langle \{x\} \rangle \cap \langle X \setminus \{x\} \rangle = \{0\} \). If \( A \) is a subgroup of \( G \), then a subset \( X \) of \( G \) is said to be independent over \( A \) if it is independent and \( \langle X \rangle \cap A = \{0\} \). The cardinality of a maximal independent set of elements of infinite order is called the torsion-free rank of \( G \), here denoted \( r_0(G) \). It is known that \( r_0(G) \) is an invariant of \( G \), i.e., all such maximal independent subsets of \( G \) have the same cardinality. It is clear that if \( h : G \to H \) is a surjective homomorphism, then \( r_0(H) \leq r_0(G) \). See \([11]\) pp. 85-86 for additional details. The torsion subgroup of an Abelian group \( G \) is denoted by \( tG \).

If \( G \) is a (not necessarily Abelian) totally bounded group, then \( \overline{G} \) denotes its (compact) Weil completion. It was shown in \([5]\) that a topological group \( G \) is pseudocompact if and only if it is \( G_\delta \)-dense in \( G \). Hence a dense subgroup of a pseudocompact group \( G \) is pseudocompact if and only if it is \( G_\delta \)-dense in \( G \).

Let \( \Lambda(G) = \{N \subseteq G : N \) is a closed, normal, \( G_\delta \)-subgroup of \( G \} \).

Now we collect some information needed later in our proof of the main result.

**Theorem 2.1.** Let \( G \) be a pseudocompact group such that \( wG > \omega \), and let \( N \subseteq \Lambda(G) \). Then

(a) \( [9, 3.3] \) \( G/N \) is compact and metrizable,
(b) \( [7, 6.2] \) \( N \) is pseudocompact, and
(c) \( [9, 2.7] \) \( wN = wG \).

**Lemma 2.2** \(([3, 2.13(b), (c)]))\). Let \( G \) be a pseudocompact group and let \( G = \bigcup_{n<\omega} A_n \), where each \( A_n \) is a subgroup of \( G \). Then there exist \( N \in \Lambda(G) \) and \( n<\omega \) such that \( A_n \cap N \) is \( G_\delta \)-dense in \( N \).

**Theorem 2.3** \(([11, 4.4], [12, 3.7.1])\). Let \( G \) be a pseudocompact Abelian group. If \( G \) contains a proper, dense pseudocompact subgroup \( H \) such that \( G/H \) can be mapped homomorphically onto some nondegenerate compact group, then \( G \) is not \( r \)-extremal.

**Theorem 2.4** \(([1, 5.7], [12, 6.4.2])\). Let \( G \) be a pseudocompact Abelian group of uncountable weight. If there exists \( N \in \Lambda(G) \) such that no connected \( M \in \Lambda(G) \) is contained in \( N \), then \( G \) is neither \( r \)-extremal nor \( s \)-extremal.

**Theorem 2.5** \(([2, 4.5], [1, 5.10], [12, 7.3])\). Let \( G \) be a pseudocompact Abelian group of uncountable weight such that \( r_0(G) > c \). Then \( G \) is neither \( r \)-extremal nor \( s \)-extremal.

3. Lemmas

In this section we collect some simple results to be used later. The technique used in the proof of Lemma 3.1 is well-known, and was used in many earlier results. See e.g., \([3, 2, 12, 1]\). For the benefit of the reader we provide the (simple) details.

(Note added September 15, 2006. The referee has pointed out that a proof of Lemma 3.1 is also available in the preprint \([10]\).)
Lemma 3.1. Let $G$ be a pseudocompact Abelian group, and let $A$ be a $G_δ$-dense subgroup of some $N \in A(G)$ such that $r_0(N/A) \geq \mathfrak{c}$. Then $G$ contains a $G_δ$-dense subgroup $H$ such that $r_0(G/H) \geq \mathfrak{c}$.

Proof. The conditions imply that there is a subset $X$ of $N \setminus A$ of elements of infinite order such that $|X| = \mathfrak{c}$ and $X$ is independent over $A$. Split $X$ into two disjoint sets $X_0$ and $X_1$, each of cardinality $\mathfrak{c}$.

By Theorem 2.1(a), the number of cosets of $N$ in $G$ is at most $\mathfrak{c}$. (In fact, either $|G/N| < \omega$ or $|G/N| = \mathfrak{c}$.) Let $\{a_\alpha + N : \alpha < \lambda\}$ be a faithful enumeration of $G/N$. We assume without loss of generality that $a_0 = 0$. By recursion on $\alpha < \lambda$ we will choose $x_\alpha \in X_0 \cup \{0\}$ such that

\[
\langle\langle x_1 \rangle\rangle \cap (\langle\langle a_\beta + x_\beta : \beta \leq \alpha \rangle\rangle + A) = \{0\}.
\]

Let $x_0 = 0$. Let $\alpha < \lambda$ and suppose that $x_\beta$ has been defined for all $\beta < \alpha$. Put $B_\alpha = \langle\langle a_\beta + x_\beta : \beta < \alpha \rangle\rangle$. Then $|B_\alpha| < \mathfrak{c}$ and $\langle\langle x_1 \rangle\rangle \cap (B_\alpha + A) = \{0\}$. Suppose that for every $x \in X_0$ we have that

\[
\langle\langle x_1 \rangle\rangle \cap (\langle\langle B_\alpha \cup \{a_\alpha + x\} \rangle\rangle + A) \neq \{0\}.
\]

Then for every $x \in X_0$ there exist $b_x \in B_\alpha$, $n_x \in \mathbb{Z}$, $p_x \in A$ and $q_x \in \langle\langle x_1 \rangle\rangle \setminus \{0\}$ such that\n
\[(\dagger) \quad q_x = b_x + n_x(a_\alpha + x) + p_x.
\]

Note (since $q_x \notin B_\alpha + A$) that no $n_x$ is equal to 0. Since $|X_0| = \mathfrak{c}$, there are distinct $x, y \in X_0$, $n \in \mathbb{Z} \setminus \{0\}$ and $b \in B_\alpha$ such that $n = n_x = n_y$ and $b = b_x = b_y$. But then by subtracting the equation $(\dagger)$ for $x$ and $y$, we get

\[n(x - y) = q_x - q_y + p_y - p_x \in \langle\langle x_1 \rangle\rangle + A,
\]

which contradicts the independence of $X$ over $A$. This completes the transfinite recursion.

Now put $B = \bigcup_{\alpha < \lambda} B_\alpha$. Then $\langle\langle x_1 \rangle\rangle \cap (B + A) = \{0\}$, hence $r_0(G/(B + A)) \geq |X_1| = \mathfrak{c}$. It is clear that $B + A$ is $G_δ$-dense in $G$.

\[\square\]

Lemma 3.2. Let $\kappa$ be an infinite cardinal. Suppose that $A$ is a family of subsets of $2^\kappa$ with the following properties:

1. If $B \in [A]^{<\kappa}$, then $\bigcap B \in A$, and
2. Each element of $A$ has cardinality $2^\kappa$.

Then there is a countably infinite family $B$ of subsets of $2^\kappa$ such that

1. $B$ is pairwise disjoint, and
2. If $A \in A$ and $B \in B$, then $|A \cap B| = 2^\kappa$.

Proof. We give $2^\kappa$ the standard Tychonov product topology. Let $V$ be the collection of all nonempty clopen subsets $V$ of $2^\kappa$ for which there is an element $A(V) \in A$ such that $|V \cap A(V)| < 2^\kappa$. Clearly, $|V| \leq \kappa$. Let $D = \{A(V) : V \in V\}$, $Y = \bigcap D$, and $V = \bigcup V$. Now $|V \cap Y| \leq |V \cap A(V)| < 2^\kappa$ for every $V \in V$, so

\[|V \cap Y| < 2^\kappa
\]

since $2^\kappa$ has cofinality at least $\kappa^+$. Then $|Y| = 2^\kappa$ by (1) and (2), hence $|Y \setminus V| = 2^\kappa$. There is consequently a countably infinite pairwise disjoint family $B$ of clopen subsets of $2^\kappa$ such that $B \cap (Y \setminus V) \neq \emptyset$ for every $B \in B$. To see that $B$ is as
required, pick arbitrary \( B \in \mathcal{B} \) and \( A \in \mathcal{A} \). If \( |B \cap A| < 2^\omega \), then \( B \in \mathcal{V} \) and hence \( B \subseteq \overline{V} \), which contradicts the fact that \( B \cap (Y \setminus \overline{V}) \neq \emptyset \). \( \square \)

**Remark 3.3.** The inclusion \( \bigcup \mathcal{B} \subseteq 2^\omega \) is necessarily proper (since \( 2^\omega \) is compact). Replacing any one element of \( \mathcal{B} \) by the complement in \( 2^\omega \) of the union of the remaining elements, we may hence assume without loss of generality that \( \mathcal{B} \) is a partition.

4. **Proof of Theorem 1.1**

We now present the proof of our main result. By Theorem 2.5, it suffices to consider groups \( G \) of torsion-free rank at most \( \omega \). Furthermore, by Theorem 2.4 we may assume that every \( N \in \Lambda(G) \) contains a connected \( M \in \Lambda(G) \). Henceforth, let \( G \) be a pseudocompact Abelian group of uncountable weight satisfying those two conditions.

**Lemma 4.1.** If \( H \) is a nontrivial connected subgroup of \( G \), then \( r_0(H) = \omega \).

**Proof.** It is clear that \( r_0(H) \leq \omega \). Let \( 0 \neq x \in H \) and let \( h \) be a continuous homomorphism from \( H \) to \( T \) such that \( h(x) \neq h(0) \). Then \( h(H) = T \) since \( H \) (and hence \( h(H) \)) is connected. It follows that \( \omega \geq r_0(H) \geq r_0(T) = \omega \), as asserted. \( \square \)

Since \( G \in \Lambda(G) \), there is a connected \( C \in \Lambda(G) \). Hence \( r_0(C) = \omega \) by Theorem 2.1(b),(c) and Lemma 4.1.

Let \( \overline{C} \) be the closure of \( C \) in \( G \). Then \( \overline{C} \) is a compact, connected group, hence is divisible [13 Theorem 24.25]. By [13 Theorem A.14] or [11 Theorem 23.1], there is a cardinal number \( \lambda \) such that \( \overline{C} \) is (algebraically) isomorphic to

\[
\bigoplus_{\alpha < \lambda} \mathbb{Q}_\alpha + t\overline{C},
\]

where each \( \mathbb{Q}_\alpha \) is a copy of the group of rational numbers \( \mathbb{Q} \). Then \( \overline{C}/t\overline{C} = \bigoplus_{\alpha < \lambda} \mathbb{Q}_\alpha \). Let \( \pi: \overline{C} \to \bigoplus_{\alpha < \lambda} \mathbb{Q}_\alpha \) be the natural homomorphism. For \( x \in \overline{C} \) let \( S(x) = \{ \alpha < \lambda : \pi(x)_\alpha \neq 0_\alpha \} \), and for \( E \subseteq \overline{C} \) let \( S(E) = \bigcup_{x \in E} S(x) \).

**Lemma 4.2.** If \( N \in \Lambda(C) \), then \( |S(\pi(N))| = \omega \).

**Proof.** Since \( N \in \Lambda(G) \), we may assume without loss of generality that \( N \) is connected. Moreover, \( N \) is nontrivial by Theorem 2.1(c). So \( r_0(N) = \omega \) by Lemma 4.1. That \( |S(\pi(N))| = \omega \) is then clear. \( \square \)

Writing \( S = S(\pi(C)) \), we have \( |S| = \omega \). Hence \( \lambda \geq \omega \), and

\[
C \subseteq \bigoplus_{\alpha \in \overline{S}} \mathbb{Q}_\alpha + \bigoplus_{\alpha \notin \overline{S}} \{0_\alpha\} + t\overline{C}.
\]

For every \( \beta \in S \), let \( \rho_\beta: \bigoplus_{\alpha \in \overline{S}} \mathbb{Q}_\alpha \to \mathbb{Q}_\beta \) be the projection.

For every nonempty \( A \subseteq \overline{S} \), put

\[
G(A) = C \cap \left( \bigoplus_{\alpha \in A} \mathbb{Q}_\alpha + \bigoplus_{\alpha \notin \overline{A}} \{0_\alpha\} + t\overline{C} \right),
\]

and let

\[
\mathcal{A} = \{ A \subseteq \overline{S} : \text{there is } N \in \Lambda(C) \text{ such that } N \subseteq G(A) \}.
\]

**Lemma 4.3.** \( \mathcal{A} \) is closed under countable intersections, and every \( A \in \mathcal{A} \) has size \( \omega \).
Proof. That \( \mathcal{A} \) is closed under countable intersections is clear, since if \( \mathcal{B} \) is any family of subsets of \( S \), then
\[
\bigcap_{B \in \mathcal{B}} G(B) = G\left(\bigcap \mathcal{B}\right)
\]
and \( \Lambda(G) \) is closed under countable intersections.

Now take an arbitrary \( A \in \mathcal{A} \). We want to prove that \( |A| = \mathfrak{c} \). Take \( N \in \Lambda(C) \) such that \( N \subseteq G(A) \). Then \( \pi(N) \subseteq A \), so \( \mathfrak{c} = |S| \geq |A| \geq |\pi(N)| = \mathfrak{c} \) by Lemma 4.2.

By Lemma 3.2 and Remark 3.3, there consequently is a (faithfully indexed) partition \( \mathcal{B} = \{B_n : n < \omega\} \) of \( S \) such that \( |B_n \cap A| = \mathfrak{c} \) for each \( B_n \in \mathcal{B} \), \( A \in \mathcal{A} \). For every \( n < \omega \), let
\[
V_n = G\left(\bigcup_{i \leq n} B_i\right).
\]
Then \( V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n \subseteq \cdots \), and \( C = \bigcup_{n < \omega} V_n \). By Theorem 2.1(b) and Lemma 2.2, there exist \( N \in \Lambda(C) \) and \( m < \omega \) such that \( H := V_m \cap N \) is \( G_\delta \)-dense in \( N \). We may assume without loss of generality that \( m = 0 \), i.e., that \( V_0 = V_0 = G(B_0) \).

Lemma 4.4. \( r_0(N/H) \geq \mathfrak{c} \).

Proof. We will prove that there is a subset \( X \) of \( N \) of cardinality \( \mathfrak{c} \) such that
\[
\begin{aligned}
(1) & \text{ each } x \in X \text{ has infinite order,} \\
(2) & \text{ } X \text{ is independent,} \\
(3) & \langle \langle X \rangle \rangle \cap H = \{0\}
\end{aligned}
\]
(hence \( \langle X \rangle \) is isomorphic to \( \bigoplus_{\alpha < \mathfrak{c}} \mathbb{Z}_\alpha \), where each \( \mathbb{Z}_\alpha \) is a copy of the group of integers \( \mathbb{Z} \)). Choose \( x_0 \in N \setminus G(B_0) \) and define \( W_0 = B_0 \). Let \( 0 < \alpha < \mathfrak{c} \) and suppose that \( x_\beta \) and \( W_\beta \) have been defined for all \( \beta < \alpha \). Then, set
\[
W_\alpha = B_0 \cup \bigcup_{\beta < \alpha} S(x_\beta),
\]
and observe that
\[
|B_1 \cap W_\alpha| = \left|B_1 \cap \bigcup_{\beta < \alpha} S(x_\beta)\right| < \mathfrak{c}.
\]
Hence \( W_\alpha \notin \mathcal{A} \), since \( B_1 \) meets every element of \( \mathcal{A} \) in a set of size \( \mathfrak{c} \), which means that \( N \notin G(W_\alpha) \); let \( x_\alpha \) be any point in \( N \setminus G(W_\alpha) \). This completes the transfinite construction.

We claim that \( X = \{x_\alpha : \alpha < \mathfrak{c}\} \) satisfies (1), (2) and (3). To prove this, let \( \alpha < \mathfrak{c} \), and let \( n \in \mathbb{Z} \setminus \{0\} \) be arbitrary. By construction we have
\[
x_\alpha \notin G\left(B_0 \cup \bigcup_{\beta < \alpha} S(x_\beta)\right),
\]
so \( S(x_\alpha) \notin B_0 \cup \bigcup_{\beta < \alpha} S(x_\beta) \); let \( \gamma \in S(x_\alpha) \) witness that relation. Then \( \rho_\gamma(\pi(x_\alpha)) \neq 0 \) and \( \rho_\gamma(\pi(x_\beta + h)) = 0 \) for every \( \beta < \alpha \) and \( h \in H \). Then clearly \( x_\alpha \) has infinite order, and
\[
x_\alpha \notin \langle \langle \{x_\beta : \beta < \alpha\} \rangle \rangle \cap H,
\]
as required. \( \square \)
Now we complete the proof of Theorem 1.1. From Lemmas 3.1 and 4.4 we conclude that $G$ has a proper dense pseudocompact subgroup $H$ such that $r_0(G/H) \geq \mathfrak{c}$. This proves that $G$ is not $s$-extremal. To see that $G$ is not $r$-extremal, note first that $r_0(G/H) \geq \mathfrak{c}$, so $G/H$ contains a subgroup isomorphic to $\bigoplus_{\alpha < \mathfrak{c}} Z_{\alpha}$. The latter can be mapped homomorphically onto $T$, and since homomorphisms into a divisible group always extend by [13, Theorem A.7], we are done by Theorem 2.3.

References


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