Cardinal invariants for κ-box products: weight, density character and Souslin number

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Abstract

The symbol $(X_I)_\kappa$ (with $\kappa \geq \omega$) denotes the space $X_I := \Pi_{i\in I} X_i$ with the κ-box topology; this has as base all sets of the form $U = \Pi_{i\in I} U_i$ with U_i open in X_i and with $|\{i \in I : U_i \neq X_i\}| < \kappa$. The symbols w, d and S denote respectively weight, density character, and Souslin number. Generalizing familiar, classical results, the authors show *inter alia*:

Theorem 3.10(b). If $\kappa \leq \alpha^+$, $|I| = \alpha$ and each X_i contains the discrete space $\{0, 1\}$ and satisfies $w(X_i) \leq \alpha$, then $w(X_\kappa) = \alpha^{<\kappa}$.

Theorem 4.17. If $\omega \leq \kappa \leq |I| \leq 2^\alpha$ and $X = (D(\alpha))^I$ with $D(\alpha)$ discrete, $|D(\alpha)| = \alpha$, then $d((X_I)_\kappa) = \alpha^{<\kappa}$.

Corollaries 5.35(a) and 5.36. Let $\alpha \geq 3$ and $\kappa \geq \omega$ be cardinals, and let $\{X_i : i \in I\}$ be a set of spaces such that $|I|^+ \geq \kappa$.

(a) If $\alpha^+ \geq \kappa$ and $\alpha \leq S(X_i) \leq \alpha^+$ for each $i \in I$ then $\alpha^{<\kappa} \leq S((X_I)_\kappa) \leq (2^\alpha)^+$; and

(b) if $\alpha^+ \leq \kappa$ and $3 \leq S(X_i) \leq \alpha^+$ for each $i \in I$ then $S((X_I)_\kappa) = (2^{<\kappa})^+$.

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1 Historical context

The most prominent, most useful, and most-studied cardinal invariants associated with topological spaces are the weight, density character, and Souslin number. Countless papers and monographs over the decades have given estimates, in some cases even precise evaluations, of the value of these invariants for the usual Tychonoff product $X_I = \Pi_{i\in I} X_i$ of a set of spaces $(X_i)_{i\in I}$ in terms of the values for the initial spaces X_i. But in the case of κ-box topologies (defined in Section 1 below) on spaces of the form X_I, very little is known, and that is fragmentary and nowhere systematically assembled.

In this paper we study with considerable thoroughness those three cardinal invariants for these modified box products, in each case seeking (as usual) estimates for the product in terms of the values for the initial spaces. Our methods are largely topological and set-theoretic, although as expected certain computations are made precise only when ZFC is enhanced with appropriate additional (consistent) axioms.

Our work draws upon, and in some cases extends, published theorems of R. Engelking and M. Karłowicz [7], W. W. Comfort and S. Negrepontis [2], [3], [4], F. S. Cater, P. Erdős and F. Galvin [1], W. W. Comfort and L. C. Robertson [5] and M. Gitik and S. Shelah [16]. We give details at appropriate points in the paper.

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2 Introduction

Hypothesized topological spaces here are not subjected to standing separation properties. Special hypotheses are imposed locally, as required.

α, β, γ and λ are cardinals, κ is an infinite cardinal, ω is the least infinite cardinal, and $\mathfrak{c}$ is the cardinality of the interval $[0,1]$. As usual, for $\alpha \geq \omega$ we write $\alpha^+ := \min\{\beta : \beta > \alpha\}$.

η and ξ are ordinals.

The symbols $w(X)$ and $d(X)$ denote respectively the weight and density character of the space X. A *cellular* family in a space X is a family of pairwise disjoint nonempty open subsets of X and $S(X)$, the *Souslin number* of X, is the cardinal number

$$\min\{\lambda : \text{no cellular family } \mathcal{A} \text{ in } X \text{ satisfies } |\mathcal{A}| = \lambda\}.$$

We here follow many authors [3], [4], though not all [8], [20], [21], in allowing w, d and S to assume finite values. If for example X is a discrete space of cardinality 17, then $w(X) = d(X) = 17$ and $S(X) = 17^+ = 18$.

For I a set, we write $[I]^\lambda := \{J \subseteq I : |J| = \lambda\}$; the notations $[I]^{<\lambda}$, $[I]^{\leq\lambda}$ are defined analogously. It is clear that if $\lambda > |I|$ then (a) $[I]^\lambda = \emptyset$ and (b) $[I]^{<\lambda}$ is the full power set $\mathcal{P}(I)$.

For a set $\{X_i : i \in I\}$ of sets we write $X_I := \Pi_{i\in I} X_i$. For $A = \Pi_{i\in I} A_i \subseteq X_I$ the *restriction set of* A is the set $R(A) := \{i \in I : A_i \neq X_i\}$. When each $X_i = (X_i, \mathcal{T}_i)$ is a space, we use the symbol $(X_I)_\kappa$ to denote X_I with the *κ-box topology*; this is the topology for which the set

$$\mathcal{U} := \{\Pi_{i\in I} U_i : U_i \in \mathcal{T}_i, |R(U)| < \kappa\}$$

is a base. (The ω-box topology on X_I, then, is the usual product topology.) We refer to $\mathcal{U}$ as *the canonical base* for $(X_I)_\kappa$, and to the elements of $\mathcal{U}$ as *canonical open sets*. By way of caution to the reader, we note that even when κ is regular, the intersection of fewer than κ-many sets, each open in $(X_I)_\kappa$, may fail to be open in $(X_I)_\kappa$. (Indeed each space X_i embeds homeomorphically as a (closed) subspace of $(X_I)_\kappa$, so if some X_i lacks that intersection property then so does $(X_I)_\kappa$.)

For simplicity we denote by the symbol **2** the discrete space of cardinality 2, and for cardinals $\alpha \geq 2$ we denote by $D(\alpha)$ the discrete space of cardinality α.

For spaces X and Y, the symbol $Y =_h X$ means that Y and X are homeomorphic; the symbol $Y \subseteq_h X$ means that X contains a homeomorphic copy of the space Y.

Definition 2.1. A cardinal κ is a *strong limit cardinal* if $\lambda < \kappa \Rightarrow 2^\lambda < \kappa$.

In 2.2–2.6 we cite the basic tools and facts we need from the elementary theory of cardinal arithmetic. For motivation, discussion and proofs where appropriate, see [3, §1], [4, Appendix A] or [19].

The familiar *beth* cardinals $\beth_\xi(\alpha)$ are defined recursively as follows.

Definition 2.2. Let $\alpha \geq 2$ be a cardinal. Then

(a) $\beth_0(\alpha) := \alpha$;

(b) $\beth_{\xi+1}(\alpha) := 2^{\beth_\xi(\alpha)}$ for each ordinal ξ; and

(c) $\beth_\xi(\alpha) := \Sigma_{\eta<\xi}\, 2^{\beth_\eta(\alpha)}$ for limit ordinals $\xi > 0$.

Remarks 2.3. Let ξ be a limit ordinal and let $\alpha \geq 2$ and $\lambda \geq \omega$ be cardinals. Then

(a) a set $S \subseteq \xi$ is cofinal in ξ if and only if $\{\beth_\eta(\alpha) : \eta \in S\}$ is cofinal in $\beth_\xi(\alpha)$; hence

(b) $\mathrm{cf}(\beth_\lambda(\alpha)) = \mathrm{cf}(\lambda)$.

Definition 2.4. For $\alpha \geq \omega$, $\log(\alpha)$ is the cardinal number

$$\log(\alpha) := \min\{\beta : 2^\beta \geq \alpha\}.$$

Notation 2.5. Let $\kappa \geq \omega$ and $\alpha \geq 2$. Then $\alpha^{<\kappa} := \Sigma_{\lambda<\kappa}\,\alpha^\lambda$.

It is well known and easy to prove that $|[\alpha]^\lambda| = \alpha^\lambda$ when $\lambda \leq \alpha$, so $|[\alpha]^{<\kappa}| = \Sigma_{\lambda<\kappa}\,\alpha^\lambda$ when $\kappa \leq \alpha^+$. For ease of reference later, we build some redundancy into the statement of Theorem 2.6.

Theorem 2.6. *Let $\alpha \geq 2$ and $\kappa \geq \omega$. Then*
(a) $\kappa \leq 2^{<\kappa} \leq \alpha^{<\kappa}$*;*
(b) *if κ is regular then $\alpha^{<\kappa} = (\alpha^{<\kappa})^{<\kappa}$;*
(c) *if κ is singular, then $(\alpha^{<\kappa})^{<\kappa} = \alpha^\kappa$;*
(d) $((\alpha^{<\kappa})^{<\kappa})^{<\kappa} = (\alpha^{<\kappa})^{<\kappa}$*.*

Remark 2.7. It is clear that the useful relation given in part (d) of Theorem 2.6 is immediate from parts (b) and (c). The authors are not acquainted with other examples in mathematics of operators which, as in Theorem 2.6(d), first stabilize at the third iteration. Responding to a request from one of us (speaking in a seminar) for terminology suitable for this phenomenon, Peter Johnstone promptly proposed the expression "sesquipotent".

The condition $(\alpha^{<\kappa})^{<\kappa} = \alpha^{<\kappa}$, satisfied by many pairs of cardinals α and κ, will play a role frequently in this paper. An alternate characterization is often useful.

Theorem 2.8. *Let $\alpha \geq 2$ and $\kappa \geq \omega$. Then*

(a) *these conditions are equivalent:*

 (i) $\alpha^{<\kappa} = (\alpha^{<\kappa})^{<\kappa}$*; and*

 (ii) *either κ is regular or there is $\nu < \kappa$ such that $\alpha^\nu = \alpha^{<\kappa}$.*

(b) *If the conditions in (a) fail, then κ and $\alpha^{<\kappa}$ are singular cardinals and $\mathrm{cf}(\alpha^{<\kappa}) = \mathrm{cf}(\kappa)$.*

Proof. (a) ((i) $\Rightarrow$ (ii)). If (ii) fails then κ is a limit cardinal and for every $\nu < \kappa$ there is a cardinal $\lambda < \kappa$ such that $\alpha^\nu < \alpha^\lambda$, so also $\alpha^{<\kappa}$ is a limit cardinal. It is easily checked that $\mathrm{cf}(\kappa) = \mathrm{cf}(\alpha^{<\kappa})$, so

$$(\alpha^{<\kappa})^{<\kappa} \geq (\alpha^{<\kappa})^{\mathrm{cf}(\kappa)} = (\alpha^{<\kappa})^{\mathrm{cf}(\alpha^{<\kappa}))} > \alpha^{<\kappa}.$$

((ii) ⇒ (i)). If κ is regular we have $(\alpha^{<\kappa})^{<\kappa} = \alpha^{<\kappa}$ by Theorem 2.6(b). Suppose then that κ is singular, hence a limit cardinal, and that there is $\nu < \kappa$ such that $\alpha^\nu = \alpha^{<\kappa}$. Then

$$\begin{aligned}(\alpha^{<\kappa})^{<\kappa} &= (\alpha^\nu)^{<\kappa} = \Sigma_{\lambda<\kappa}\,(\alpha^\nu)^\lambda = \Sigma_{\nu<\lambda<\kappa}\,(\alpha^\nu)^\lambda\\ &= \Sigma_{\nu<\lambda<\kappa}\,\alpha^\lambda = \alpha^{<\kappa}.\end{aligned}$$

(b) Clearly κ is singular, $\alpha^{<\kappa}$ is limit, and

$$\mathrm{cf}(\alpha^{<\kappa}) = \mathrm{cf}(\kappa) < \kappa \leq \alpha^{<\kappa}.$$

Hence $\alpha^{<\kappa}$ is singular. □

Remarks 2.9. (a) As our title and our Abstract indicate, we are concerned here with the weight, density character, and Souslin number of (sometimes specialized) products of the form $(X_I)_\kappa$; the corresponding results are contained in Sections 2, 3 and 4, respectively.

(b) As the reader knows well, the "functions" w, d and S enjoy specific useful monotonicity properties; we have in mind these familiar phenomena:

(*i*) If X and Y are spaces and $Y \subseteq_h X$, then $w(Y) \leq w(X)$;

(*ii*) If $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on a set X with $\mathcal{T}_1 \subseteq \mathcal{T}_2$, then $d(X, \mathcal{T}_1) \leq d(X, \mathcal{T}_2)$ and $S(X, \mathcal{T}_1) \leq S(X, \mathcal{T}_2)$.

On the other hand, both the analogue of (*i*) for d and S and of (*ii*) for w can fail. For example, with $X = \mathbf{2}^{\mathfrak{c}}$ and

$$Y := \{x \in X : |\{i \in I : x_i \neq 0\}| = 1\},$$

one has Y discrete in X with $|Y| = \mathfrak{c}$ and $d(X) = \omega < \mathfrak{c} = d(Y)$, also $S(X) = \omega^+ < \mathfrak{c}^+ = S(Y)$. And with $X = \mathbf{2}^{\mathfrak{c}}$ and Y' a countable dense subset of X one has $w(Y') = w(X) = \mathfrak{c}$ when the usual product topology $\mathcal{T}_1$ is considered, but $w(Y', \mathcal{T}_2) = \omega < \mathfrak{c}$ when Y' is given the discrete topology $\mathcal{T}_2 \supseteq \mathcal{T}_1$.

We use the indicated monotonicity properties (*i*) and (*ii*) frequently in this paper, without warning or comment. We use also the fact that if X is a space and Y is dense in X, then necessarily $S(Y) = S(X)$.

3 On the weight of κ-box products

Discussion 3.1. It is well known [8, 2.3.F(a)] for each set $\{X_i : i \in I\}$ of T_1-spaces with $w(X_i) \geq 2$ that $X_I := \Pi_{i\in I}\,X_i$ satisfies

$$w(X_I) = \max\{\sup_{i\in I}\, w(X_i), |I|\}.$$

In particular,

$$w(X_I) = |I| \text{ if each } X_i \text{ satisfies } w(X_i) \leq |I|. \tag{3.1}$$

In Theorem 3.10 we give the correct analogue of (3.1) for κ-box topologies.

Lemma 3.2. *Let $\alpha \geq \omega$ and $\kappa \geq \omega$. Then*

$$w((\mathbf{2}^\alpha)_\kappa) \geq \alpha.$$

Proof. Let $Y \subseteq X = \mathbf{2}^\alpha$ be as in Remarks 2.9(b). Then Y is discrete in $\mathbf{2}^\alpha$, hence is discrete in $(\mathbf{2}^\alpha)_\kappa$, so

$$w((\mathbf{2}^\alpha)_\kappa) \geq w(Y) = \alpha.$$

□

Theorem 3.3. *Let $\alpha \geq \omega$ and $\kappa \geq \omega$ and let $\{X_i : i \in I\}$ be a set of spaces such that $w(X_i) \leq \alpha$ for each $i \in I$. Then*

(a) $\sup_{i\in I} w(X_i) \leq w((X_I)_\kappa) \leq \alpha^{<\kappa} \cdot |I|^{<\kappa}$; *and*

(b) *if in addition $\mathbf{2} \subseteq_h X_i$ for each $i \in I$, then also $w((X_I)_\kappa) \geq |I|$.*

Proof. (a) Let $\mathcal{B}_i$ be a base for X_i with $|\mathcal{B}_i| \leq \alpha$ and with $X_i \in \mathcal{B}_i$, and for $\lambda < \kappa$ let

$$\mathcal{B}(\lambda) := \{B = \Pi_{i\in I} B_i : B_i \in \mathcal{B}_i, |R(B)| = \lambda\}.$$

Then $|\mathcal{B}(\lambda)| \leq |[I]|^\lambda \cdot \alpha^\lambda$, and since $\mathcal{B} := \bigcup_{\lambda<\kappa} \mathcal{B}(\lambda)$ is a base for $(X_I)_\kappa$ we have

$$w((X_I)_\kappa) \leq |\mathcal{B}| \leq \Sigma_{\lambda<\kappa} |[I]^\lambda| \cdot \alpha^\lambda = |I|^{<\kappa} \cdot \alpha^{<\kappa}.$$

Since $X_i \subseteq_h (X_I)_\kappa$, we have $w((X_I)_\kappa) \geq w(X_i)$ for each $i \in I$. Hence $w((X_I)_\kappa) \geq \sup_{i\in I} w(X_i)$.

(b) It follows from $\mathbf{2} \subseteq_h X_i$ that $\mathbf{2}^I \subseteq_h X$ and from Lemma 3.2 we have $w((X_I)_\kappa) \geq w((\mathbf{2}^I)_\kappa) \geq |I|$. □

For future reference we re-state this portion of Theorem 3.3.

Corollary 3.4. *Let α and κ be infinite cardinals and let $\{X_i : i \in I\}$ be a set of spaces such that $|I| \leq \alpha$ and $w(X_i) \leq \alpha$ for each $i \in I$. Then $w((X_I)_\kappa) \leq \alpha^{<\kappa}$.*

Discussion 3.5. If $\omega \leq \alpha < \alpha^+ < \kappa$, then $w((\mathbf{2}^\alpha)_\kappa) = 2^\alpha$, while $\alpha^{<\kappa} \geq \alpha^{(\alpha^+)} = 2^{(\alpha^+)}$. In many models of set theory and for many cardinals α one has $2^{(\alpha^+)} > 2^\alpha$, and in such cases the inequality $w((\mathbf{2}^\alpha)_\kappa) \leq \alpha^{<\kappa}$ of Corollary 3.4 becomes strict. That explains why the formula $w(X_\kappa) = \alpha^{<\kappa}$ cannot be asserted without restraint in Corollary 3.4, even when $|I| = \alpha$. Our next goal in this section is to show that, subject only to the simple restrictions $\kappa \leq \alpha^+$ and $|I| = \alpha$, the inequality $w((X_I)_\kappa) \leq \alpha^{<\kappa}$ of Corollary 3.4 becomes an equality (Theorem 3.10).

Lemma 3.6. *Let α and κ be infinite cardinals such that $\kappa \leq \alpha^+$. Then*
(a) *if $\lambda < \kappa$ and $\lambda \leq \alpha$, then $w((\mathbf{2}^\alpha)_\kappa) \geq 2^\lambda$; and*
(b) $w((\mathbf{2}^\alpha)_\kappa) \geq 2^{<\kappa}$.

Proof. (a) If $\kappa = \alpha^+$ then $(\mathbf{2}^\alpha)_\kappa$ is the discrete space $D(2^\alpha)$, which has weight $2^\alpha = 2^{<\kappa} \geq 2^\lambda$. We assume therefore that $\kappa \leq \alpha$. The space $(\mathbf{2}^\lambda)_\kappa$ is then homeomorphic to a discrete subspace of $(\mathbf{2}^\alpha)_\kappa$, so $w((\mathbf{2}^\alpha)_\kappa) \geq w((\mathbf{2}^\lambda)_\kappa) = |\mathbf{2}^\lambda| = 2^\lambda$.

(b) is immediate from (a). □

Theorem 3.7. *Let κ and α be infinite cardinals. Then*
(a) *if $\kappa \geq \alpha^+$ then $w((\mathbf{2}^\alpha)_\kappa) = 2^\alpha$;*
(b) *if $\kappa \leq \alpha^+$ then $w((\mathbf{2}^\alpha)_\kappa) = \alpha^{<\kappa}$.*

Proof. (a) is obvious, since $(\mathbf{2}^\alpha)_\kappa$ is discrete.

(b) The inequality $\leq$ is given by Corollary 3.4. We show $\geq$.

If $\kappa = \alpha^+$ then $(\mathbf{2}^\alpha)_\kappa$ is the discrete space $D(2^\alpha)$, which has weight $2^\alpha = \alpha^\alpha = \alpha^{<\kappa}$. We assume in what follows that $\kappa \leq \alpha$ and we consider two cases.

Case 1. $2^{<\kappa} \leq \alpha$.

If $w((\mathbf{2}^\alpha)_\kappa) \geq \alpha^{<\kappa}$ fails then there is $\lambda < \kappa$ such that $w((\mathbf{2}^\alpha)_\kappa) < \alpha^\lambda$; we fix such λ and we choose in $(\mathbf{2}^\alpha)_\kappa$ a base $\mathcal{B}$ of canonical open sets such that $|\mathcal{B}| < \alpha^\lambda$. From Lemma 3.6(b) we have $|\mathcal{B}| \geq 2^{<\kappa}$.

For every $A \in [\alpha]^{<\kappa}$ there is $B \in \mathcal{B}$ such that $A \subseteq R(B)$. (To check that, it is enough to choose in $(\mathbf{2}^\alpha)_\kappa$ a canonical open set $U = \Pi_{i\in I} U_i$ with $R(U) = A$ and $x \in U$, and then to find $B \in \mathcal{B}$ such that $x \in B \subseteq U$. Then B is as required.) Thus

$$[\alpha]^{<\kappa} \subseteq \bigcup\{[R(B)]^{<\kappa} : B \in \mathcal{B}\}. \tag{3.2}$$

For each $B \in \mathcal{B}$ we have $|R(B)| < \kappa$ and hence $[R(B)]^{<\kappa} = \mathcal{P}(R(B))$. Therefore $|[R(B)]^{<\kappa}| = 2^{|R(B)|} \leq 2^{<\kappa}$. From (3.2), then, we have the contradiction

$$\alpha^{<\kappa} = |[\alpha]^{<\kappa}| \leq \Sigma\{[R(B)]^{<\kappa}| : B \in \mathcal{B}\} \leq$$
$$2^{<\kappa} \cdot |\mathcal{B}| = |\mathcal{B}| < \alpha^\lambda \leq \alpha^{<\kappa}.$$

Case 2. Case 1 fails.

Then there is $\lambda < \kappa$ such that $2^\lambda > \alpha$. If the desired inequality $w((\mathbf{2}^\alpha)_\kappa) \geq \alpha^{<\kappa}$ fails then there is $\mu < \kappa$ such that $w((\mathbf{2}^\alpha)_\kappa) < \alpha^\mu$, and then with $\delta := \max\{\lambda, \mu\}$ we have the contradiction

$$w((\mathbf{2}^\alpha)_\kappa) < \alpha^\delta \leq (2^\delta)^\delta = 2^\delta = w((\mathbf{2}^\delta)_\kappa) \leq w((\mathbf{2}^\alpha)_\kappa).$$

□

Remark 3.8. When $\kappa = \alpha^+$ in Theorem 3.7, parts (a) and (b) are compatible since $\alpha^{<\kappa} = \alpha^\alpha = 2^\alpha$ in that case.

Corollary 3.9. *Let κ and α be infinite cardinals. Then*

$$w((\mathbf{2}^{(\alpha^{<\kappa})})_\kappa) = (\alpha^{<\kappa})^{<\kappa} = w((\mathbf{2}^{((\alpha^{<\kappa})^{<\kappa})})_\kappa).$$

Proof. From Theorem 2.6(a) we have $\kappa \leq \alpha^{<\kappa} \leq (\alpha^{<\kappa})^{<\kappa}$. The first equality then results by replacing α by $\alpha^{<\kappa}$ in Theorem 3.7(b), the second equality results by making the same substitution one more time. □

Theorem 3.10. *Let α and κ be infinite cardinals, and let $\{X_i : i \in I\}$ be a set of spaces such that $|I| = \alpha$, and $\mathbf{2} \subseteq_h X_i$ and $w(X_i) \leq \alpha$ for each $i \in I$. Then*

(a) *if $\kappa \leq \alpha^+$ then $w((X_I)_\kappa) = \alpha^{<\kappa}$;*
(b) *if $\kappa \geq \alpha^+$ then $w((X_I)_\kappa) = 2^\alpha$.*

Proof. We have $\mathbf{2}^\alpha \subseteq_h X$ and hence $(\mathbf{2}^\alpha)_\kappa \subseteq_h (X_I)_\kappa$. Then from Theorem 3.7 and Corollary 3.4 it follows that

$$\alpha^{<\kappa} = w((\mathbf{2}^\alpha)_\kappa) \leq w((X_I)_\kappa) \leq \alpha^{<\kappa}$$

in (a), and

$$2^\alpha = w((\mathbf{2}^\alpha)_\kappa) \leq w((X_I)_\kappa) \leq \alpha^{<\kappa} \leq \alpha^\alpha = 2^\alpha$$

in (b). □

Corollary 3.11. *Let α and κ be infinite cardinals, and let $\{X_i : i \in I\}$ be a set of spaces such that $\mathbf{2} \subseteq_h X_i$ for each $i \in I$. If $w(X_i) \leq (\alpha^{<\kappa})^{<\kappa}$ for each $i \in I$ and $\alpha^{<\kappa} \leq |I| \leq (\alpha^{<\kappa})^{<\kappa}$, then $w((X_I)_\kappa) = (\alpha^{<\kappa})^{<\kappa}$.*

Proof. For $\leq$, replace α by $(\alpha^{<\kappa})^{<\kappa}$ in Theorem 3.3(a) and use Theorem 2.6(d).

For $\geq$, it is enough to note from Corollary 3.9 that

$$w((X_I)_\kappa) \geq w((\mathbf{2}^I)_\kappa) \geq w((\mathbf{2}^{(\alpha^{<\kappa})})_\kappa) = (\alpha^{<\kappa})^{<\kappa}.$$

□

Like the authors, the reader will have noted already at this stage a distinction in kind between the pleasing, clear-cut result given in Discussion 3.1 concerning the weight of a product in the usual product topology and the less satisfactory statement given in Corollary 3.11; in this latter, the weight of spaces of the form $(\mathbf{2}^I)_\kappa$ is determined by $|I|$, but unexpectedly such products which differ in size may have the same weight.

Corollary 3.12. *Let α, κ and λ be infinite cardinals such that $\lambda \leq \kappa$, and let $\{X_i : i \in I\}$ be a set of spaces such that $|I| = \alpha$, and $\mathbf{2} \subseteq_h X_i$ and $w(X_i) \leq \alpha$ for each $i \in I$. Then $w((X_I)_\lambda) \leq w((X_I)_\kappa)$.*

Proof. Necessarily we have $\lambda \leq \kappa \leq \alpha^+$, or $\lambda \leq \alpha^+ \leq \kappa$, or $\alpha^+ \leq \lambda \leq \kappa$. In those three cases, Theorem 3.10 gives respectively

$w((X_I)_\lambda) = \alpha^{<\lambda} \leq \alpha^{<\kappa} = w((X_I)_\kappa)$,
$w((X_I)_\lambda) = \alpha^{<\lambda} \leq \alpha^\alpha = 2^\alpha = w((X_I)_\kappa)$, and
$w((X_I)_\lambda) = 2^\alpha = w((X_I)_\kappa)$. □

Remarks 3.13. (a) Surely Corollary 3.12 is as expected. Presumably a short, direct proof is available but the authors' search for that was unsuccessful. We note however that, as the simple example in Remark 2.9(b)(*ii*) shows, a larger topology (for example, the discrete topology) on a given set may have a strictly smaller weight than does a smaller Tychonoff topology.

(b) The authors find surprising both the extent of validity of the formula given in Theorem 3.10 and the simplicity of its proof. We had anticipated finding an explicit formula for $w((X_I)_\kappa)$ only under special axioms and assumptions (perhaps GCH, for example), and we had anticipated the necessity to consider, at the least, such cardinals as $\mathrm{cf}(\kappa)$, $\mathrm{cf}(\alpha)$ and $\log(\alpha)$, as well as the least cardinal γ such that $\alpha^\gamma > \alpha$.

4 On the density character of κ-box products

In this section we continue to investigate spaces of the form $(X_I)_\kappa$, focusing now on the invariant d rather than on w. Our point of departure and motivation is the paradigmatic trilogy of Theorems 4.1, 4.2 and 4.3, which for the usual product topology give respectively upper bounds, lower bounds, and conditions of equality for (certain) numbers of the form $d((X_I)_\kappa)$. To avoid unnecessary restrictions, we state these three familiar results in considerable generality. Standard treatments often impose stronger separation properties according to authors' conventions, but the published proofs (of

Theorems 4.2 and 4.3 in [3, 3.19 and 3.20], for example) suffice to establish Theorems 4.1–4.3 in the form we have chosen. Theorem 4.1 is, of course, the classic theorem of Hewitt, Marczewski and Pondiczery [17], [25], [26], stated here in two useful equivalent forms; and Theorem 4.2 is its converse.

Our κ-box analogues to Theorems 4.1 and 4.2 are given in 4.7–4.11 and 4.22–4.23, respectively. The quest for the exact κ-box analogue of Theorem 4.3—that is, the search for a specific cardinal number δ depending on the variables $|I|$, $d(X_i)$ $(i \in I)$ and κ so that $d((X_I)_\kappa) = \delta$—is elusive, perhaps unattainable. For example, answering a question from [2], [3], Cater, Erdös, and Galvin [1] have shown that in some models for $\beta = \aleph_\omega$ the inequalities

$$d((\mathbf{2}^{(\beta^+)})_{\omega^+}) = \mathfrak{c} < \beta = \log(2^\beta) < d((\mathbf{2}^{(2^\beta)})_{\omega^+})$$

occur. Furthermore, it has been known for some time [1], [5] that consistently $d((\mathbf{2}^\beta)_{\omega^+}) = (\log(\beta))^\omega$ for every infinite cardinal β. The question whether that equality holds in (all models of) ZFC, raised in [1], was answered in the negative by Gitik and Shelah [16]; we discuss their models in 4.14(d)–(g).

The foregoing paragraph explains why we are able for $\kappa > \omega$ to offer exact computations of the form $d((X_I)_\kappa) = \delta$, in parallel with Theorem 4.3, only for spaces X_i $(i \in I)$ and κ subject to severe constraints. Our (few) contributions of this sort are given in Corollary 4.9(a) and Theorems 4.17—4.20 below.

Theorem 4.1. [Version 1] *Let $\alpha \geq \omega$, $X_I = \Pi_{i\in I} X_i$ with $d(X_i) \leq \alpha$ for each $i \in I$ and with $|I| \leq 2^\alpha$. Then $d(X_I) \leq \alpha$.*

[Version 2] *Let I be an infinite set and $\{X_i : i \in I\}$ a set of spaces. Then $d(X_I) \leq \max\{\sup\{d(X_i) : i \in I\}, \log|I|\}$.*

Theorem 4.2. *Let $\alpha \geq \omega$ and let $X_I = \Pi_{i\in I} X_i$ with $S(X_i) \geq 3$ for each $i \in I$. If $d(X_i) > \alpha$ for some $i \in I$, or if $|I| > 2^\alpha$, then $d(X_I) > \alpha$.*

Theorem 4.3. *If $\{X_i : i \in I\}$ is a family of spaces such that $S(X_i) \geq 3$ for each $i \in I$ and $|I| \geq \omega$, then*

$$d(X_I) = \max\{\sup\{d(X_i) : i \in I\}, \log|I|\}.$$

Part A. Upper Bounds for $d((X_I)_\kappa)$.

We say that a subset A of a space X is *strongly discrete* (in X) if there is a family $\{U(a) : a \in A\}$ of pairwise disjoint open subsets of X such that $a \in U(a)$ for each $a \in A$. Simple examples show that a strongly discrete set need not be closed. It is clear, however, that if κ is fixed and every $A \in [X]^{<\kappa}$ is strongly discrete, then also every $A \in [X]^{<\kappa}$ is closed in X. That motivates the following terminology.

Definition 4.4. Let $\kappa \geq \omega$ and let X be a space. Then X is *strongly κ-discrete* if every $A \in [X]^{<\kappa}$ is strongly discrete.

Remarks 4.5. (a) The terminology in Definition 4.4 is not in universal usage. Note that the separation requirement applies only to sets $A \in [X]^{<\kappa}$, not to all $A \in [X]^{\leq\kappa}$. Note also that since in a strongly κ-discrete space X each set $A \in [X]^{<\kappa}$ is both closed and discrete, the condition is strictly stronger than the condition that each discrete set $A \in [X]^{<\kappa}$ is strongly discrete.

(b) We note that a space which for some $\kappa \geq \omega$ is strongly κ-discrete is a Hausdorff space.

(c) We give the following lemma in the generality it warrants, but in fact we will use it only when each of the spaces E_i is discrete.

Lemma 4.6. *Let $\kappa \geq \omega$ and let $E = E_I = \Pi_{i\in I} E_i$ with each space E_i strongly κ-discrete. Then $(E_I)_\kappa$ is strongly κ-discrete.*

Proof. Given $A \in [E]^{<\kappa}$ there is $J \in [I]^{<\kappa}$ such that the projection $\pi_J : E \twoheadrightarrow \Pi_{i\in J} E_i$, when restricted to A, is an injection. (If $\kappa > |I|$ we may take $J = I \in [I]^{<\kappa}$.) Now for $i \in I$ and $a \in A$ we choose a neighborhood $U_i(a)$ of a_i in X_i such that

(a) $U_i(a_i) = U_i(b_i)$ if $a, b \in A$ and $a_i = b_i$, and

(b) $U_i(a_i) \cap U_i(b_i) = \emptyset$ if $a, b \in A$ and $a_i \neq b_i$.

[Such a family $\{U_i(a_i) : a \in A\}$ exists in X_i since $\pi_i[A] \in [X_i]^{<\kappa}$.] Then the sets $U(a) := (\Pi_{i\in J} U_i(a_i)) \times \Pi_{i\in I\setminus J} E_i$ are open in $(E_I)_\kappa$ and are pairwise disjoint with $a \in U(a)$ for each $a \in A$. □

The principal result of this section is given in Theorem 4.8. The following lemma does most of the work.

Lemma 4.7. *Let $\alpha \geq 2$, $\beta \geq \omega$, and $\kappa \geq \omega$ be cardinals and let $E := (D(\alpha))^{2^\beta}$. Then $d(E_\kappa) \leq \alpha^{<\kappa} \cdot (\beta^{<\kappa})^{<\kappa}$.*

Proof. We set $X = \mathbf{2}^\beta$. Since $w(\mathbf{2}) = 2 \leq \beta$ there is (by Corollary 3.4) a base $\mathcal{B}$ for $(\mathbf{2}^\beta)_\kappa$ such that $|\mathcal{B}| = \beta^{<\kappa}$. We assume without loss of generality that the elements of $\mathcal{B}$ are drawn from the canonical base for $(\mathbf{2}^\beta)_\kappa$ (see [8, 1.1.15]). Let $\mathbb{C} := \{\mathcal{C} \subseteq \mathcal{P}(\mathcal{B}) : \mathcal{C}$ is cellular in $(\mathbf{2}^\beta)_\kappa$ and $|\mathcal{C}| < \kappa\}$, and for each $\mathcal{C} \in \mathbb{C}$ and $f : \mathcal{C} \to D(\alpha)$ define $p(\mathcal{C}, f) \in E = (D(\alpha))^{\mathbf{2}^\beta}$ by

$$(p(\mathcal{C}, f))_x = \left\{ \begin{array}{c} f(x) \text{ if there is } C \in \mathcal{C} \text{ such that } x \in C \\ 0 \text{ if } x \in \mathbf{2}^\beta \backslash \bigcup \mathcal{C} \end{array} \right\}.$$

We set $A := \{p(\mathcal{C}, f) : \mathcal{C} \in \mathbb{C}, f : \mathcal{C} \to D(\alpha)\}$.

Since $\kappa \leq \beta^{<\kappa} = |\mathcal{B}|$ we have $|[\mathcal{B}]^{<\kappa}| = (\beta^{<\kappa})^{<\kappa}$.

Then since $|\mathbb{C}| \leq |[\mathcal{B}]^{<\kappa}| = (\beta^{<\kappa})^{<\kappa}$ and for $\mathcal{C} \in \mathbb{C}$ we have $|\alpha^{\mathcal{C}}| = \alpha^{|\mathcal{C}|} \leq \alpha^{<\kappa}$-many functions $f : \mathcal{C} \to D(\alpha)$, it follows that

$$|A| \leq \alpha^{<\kappa} \cdot (\beta^{<\kappa})^{<\kappa}.$$

It suffices then to show that A is dense in $E_\kappa = ((D(\alpha))^{2^\beta})_\kappa$.

Let $U = \Pi_{x \in \mathbf{2}^\beta} U_x$ be a canonical open subset of E_κ. Without loss of generality we take $|U_x| = 1$ when $x \in R(U) \in [\mathbf{2}^\beta]^{<\kappa}$ (and necessarily $U_x = D(\alpha)$ when $x \in \mathbf{2}^\beta \backslash R(U)$). We define $f : R(U) \to D(\alpha)$ so that $U_x = \{f(x)\}$. Since $(\mathbf{2}^\beta)_\kappa$ is strongly κ-discrete (by Lemma 4.6) and $R(U) \in [\mathbf{2}^\beta]^{<\kappa}$, there is a family $\mathcal{C} = \{C(x) : x \in R(U)\} \in \mathbb{C}$ of pairwise disjoint open subsets of $(\mathbf{2}^\beta)_\kappa$ such that $x \in C(x)$ for each $x \in R(U)$. Then for $x \in R(U)$ we have $x \in C(x) \in \mathcal{C} \in \mathbb{C}$, and $(p(\mathcal{C}, f))_x = f(x) \in U_x$; it follows that $p(\mathcal{C}, f) \in A \cap U$, as required. □

The proof of Lemma 4.7 seemed so natural that for some time we considered its statement to be optimal. However, a stronger statement is available. This is the principal result of this section, given now in two equivalent formulations.

Theorem 4.8. *Let $\alpha \geq 2$, $\beta \geq \omega$, and $\kappa \geq \omega$ be cardinals.*

(a) *Let $E := (D(\alpha))^{2^\beta}$. Then $d(E_\kappa) \leq (\alpha \cdot \beta)^{<\kappa}$.*

(b) *Let $E := (D(\alpha))^\beta$. Then $d(E_\kappa) \leq (\alpha \cdot \log(\beta))^{<\kappa}$.*

Proof. [When (a) is known, (b) follows upon replacing 2^β in (a) by β and using the inequality $\beta \leq 2^{\log(\beta)}$. To derive (a) from (b), replace β in (b) by 2^β and use that $\log(2^\beta) \leq \beta$.]

To prove (a), we consider two cases.

Case 1. κ is regular. Then it follows from Lemma 4.7 and Theorem 2.6(b) that

$$d(E_\kappa) \leq \alpha^{<\kappa} \cdot (\beta^{<\kappa})^{<\kappa} = \alpha^{<\kappa} \cdot \beta^{<\kappa} = (\alpha \cdot \beta)^{<\kappa}.$$

Case 2. κ is singular (hence, a limit cardinal).

[Here we use a trick taken from [1, p. 308].] For $\lambda < \kappa$ there is, by Case 1 applied to the regular cardinal λ^+, a dense set $A(\lambda) \subseteq E_{\lambda^+}$ such that $|A(\lambda)| \leq (\alpha\cdot\beta)^{<\lambda^+} = (\alpha\cdot\beta)^\lambda$. The set $A := \bigcup_{\lambda<\kappa} A(\lambda)$ is clearly dense in E_κ, so

$$d(E_\kappa) \leq |A| \leq \Sigma_{\lambda<\kappa}\,(\alpha\cdot\beta)^\lambda = (\alpha\cdot\beta)^{<\kappa}.$$

□

Corollary 4.9. *Let $\alpha \geq 2$ and $\kappa \geq \omega$ be cardinals, and let $1 \leq \lambda \leq (\alpha^{<\kappa})^{<\kappa}$ and $1 \leq \mu \leq 2^{((\alpha^{<\kappa})^{<\kappa})}$. Then*
(a) $d(((D((\alpha^{<\kappa})^{<\kappa}))^{2^{((\alpha^{<\kappa})^{<\kappa})}})_\kappa) = (\alpha^{<\kappa})^{<\kappa}$*; and*
(b) $d(((D(\lambda))^\mu)_\kappa) \leq (\alpha^{<\kappa})^{<\kappa}$.

Proof. Clearly (b) is immediate from (a). To prove (a), it is enough to replace α and β in Theorem 4.8 by $(\alpha^{<\kappa})^{<\kappa}$ and then to use Theorem 2.6(d). □

Discussion 4.10. A convenient method of proof of the Hewitt-Marczewski-Pondiczery theorem (Theorem 4.1), adopted by many expositors, is to prove first that the tractable space $E := (D(\alpha))^{2^\alpha}$ has a dense subset A with $|A| = \alpha$; since evidently there is a continuous function f from E onto a dense subset of X, the set $f[A]$ is dense in X, with $|f[A]| \leq |A| = \alpha$. The identical argument suffices to derive Corollary 4.11 from Theorem 4.8 and Corollary 4.9.

Corollary 4.11. *Let $\alpha \geq 2$, $\beta \geq \omega$ and $\kappa \geq \omega$ be cardinals and let $\{X_i : i \in I\}$ be a set of spaces.*

(a) If $d(X_i) \leq \alpha$ for each $i \in I$ and $|I| \leq 2^\beta$, then $d((X_I)_\kappa) \leq (\alpha\cdot\beta)^{<\kappa}$;

(b) if $d(X_i) \leq \alpha$ for each $i \in I$ and $|I| \leq \beta$, then $d((X_I)_\kappa) \leq (\alpha\cdot\log(\beta))^{<\kappa}$; and

(c) if $d(X_i) \leq (\alpha^{<\kappa})^{<\kappa}$ for each $i \in I$ and $|I| \leq 2^{((\alpha^{<\kappa})^{<\kappa})}$, then $d((X_I)_\kappa) \leq (\alpha^{<\kappa})^{<\kappa}$.

Remark 4.12. For cardinals α and κ such that $\alpha^{<\kappa} = \alpha = \beta$, Corollary 4.11(a) is [3, 3.18] and was also mentioned in [4, p. 76].

We restate Corollary 4.11(b) in the form most easily comparable with Version 2 of Theorem 4.1.

Theorem 4.13. *Let $\{X_i : i \in I\}$ be a set of spaces with each $d(X_i) = \alpha_i$, and let $\alpha := \sup_{i\in I}\,\alpha_i$. Then*

$$d((X_I)_\kappa) \leq \max\{\alpha^{<\kappa}, (\log(|I|))^{<\kappa}\}.$$

As we see in Discussion 4.14(d), however, the inequality in Theorem 4.13 can be strict. Thus consistently the obvious κ-box analogue of Theorem 4.3 can fail.

Discussion 4.14. The two results

$$d(((D(\alpha))^{2^\alpha})_\kappa) \leq \alpha^{<\kappa}$$

and

$$d(((D((\alpha^{<\kappa})^{<\kappa}))^{2^{((\alpha^{<\kappa})^{<\kappa})}})_\kappa) = (\alpha^{<\kappa})^{<\kappa},$$

valid for $\alpha \geq 2$ and $\kappa \geq \omega$ and given by Theorem 4.8(a) and Corollary 4.9(a) respectively, suggest the attractive "intermediate" speculation

$$d(((D(\alpha^{<\kappa}))^{2^{(\alpha^{<\kappa})}})_\kappa) \leq \alpha^{<\kappa} \tag{4.1}$$

which, if valid, would yield these two weaker statements:

$$d(((D(\alpha))^{2^{(\alpha^{<\kappa})}})_\kappa) \leq \alpha^{<\kappa} \tag{4.2}$$

and

$$d(((D(\alpha^{<\kappa}))^{2^\alpha})_\kappa) \leq \alpha^{<\kappa}. \tag{4.3}$$

We discuss what we do and do not know about the truth value of (4.1), (4.2), and (4.3).

(a) (4.1) (*hence also* (4.2) *and* (4.3)) *holds for all* α *and* κ *satisfying* $(\alpha^{<\kappa})^{<\kappa} = \alpha^{<\kappa}$. This is obvious from Corollary 4.9(a). Thus by Theorem 2.8(a) the conditions (4.1), (4.2) and (4.3) hold (for all $\alpha \geq 2$) when κ is regular or there is $\nu < \kappa$ such that $\alpha^\nu = \alpha^{<\kappa}$.

(b) (4.3) *holds in* ZFC, *for all* $\kappa \geq \omega$, *when* $2 \leq \alpha < \omega$. This is obvious, since $|D(\alpha^{<\kappa})^{2^\alpha}| = \alpha^{<\kappa}$ in that case.

(c) *for all* $\alpha \geq \kappa$, (4.3) *fails* (*hence* (4.1) *fails*) *in* ZFC *for certain* κ. In fact, we prove this statement:

Let $\alpha \geq \omega$. *There are arbitrarily large cardinals* κ *such that the space* $E := (D(\alpha^{<\kappa}))^\alpha$ *satisfies* $d(E_\kappa) > \alpha^{<\kappa}$.

To prove that, choose $\lambda \geq \omega$ such that $\mathrm{cf}(\lambda) \leq \alpha$ (for example, set $\lambda := \omega$). Then, set $\kappa := \beth_\lambda(\alpha)$. Since $\kappa > \alpha$ the space E_κ is discrete, so from Remarks 2.3(b) we have

$$d(E_\kappa) = |E| = \kappa^\alpha \geq \kappa^{\mathrm{cf}(\lambda)} = \kappa^{\mathrm{cf}(\kappa)} > \kappa = \alpha^{<\kappa}.$$

(d) *Consistently,* (4.2) *fails* (*hence* (4.1) *fails*) *when* $\alpha = 2$ *and* $\kappa = \aleph_1$. Indeed, Gitik and Shelah [16], answering a question left unresolved in [1] and [5], have constructed models $\mathbb{V}_1$ and $\mathbb{V}_2$ of ZFC such that

$$d((\mathbf{2}^{\aleph_\omega})_{\aleph_1}) = \begin{cases} \aleph_{\omega+1} & \text{in} \quad \mathbb{V}_1 \\ \aleph_{\omega+2} & \text{in} \quad \mathbb{V}_2 \end{cases},$$

with $2^{\aleph_\omega} = \aleph_\omega^\omega = \aleph_{\omega+2}$ in each case and with "GCH below $\aleph_\omega$", so that $2^{<\aleph_\omega} = \aleph_\omega$. Then, taking $\alpha = 2$ and $\kappa = \aleph_\omega$ in Theorem 4.8(a) we have

$$2^{\aleph_\omega} \geq d((\mathbf{2}^{(2^{(2^{<\aleph_\omega})})})_{\aleph_\omega}) \geq d((\mathbf{2}^{\aleph_\omega})_{\aleph_1}) = \aleph_{\omega+2} = 2^{\aleph_\omega} > \aleph_\omega = 2^{<\aleph_\omega}$$

in the Gitik-Shelah model $\mathbb{V}_2$, while in $\mathbb{V}_1$ we have

$$2^{\aleph_\omega} \geq d((\mathbf{2}^{(2^{(2^{<\aleph_\omega})})})_{\aleph_\omega}) \geq d((\mathbf{2}^{\aleph_\omega})_{\aleph_1}) = \aleph_{\omega+1} > \aleph_\omega = 2^{<\aleph_\omega}.$$

Thus in both $\mathbb{V}_1$ and $\mathbb{V}_2$ we have

$$2^{\aleph_\omega} \geq d((\mathbf{2}^{(2^{(2^{<\aleph_\omega})})})_{\aleph_\omega}) > 2^{<\aleph_\omega},$$

so (4.2) (hence (4.1)) fails there.

(e) We interpret the cited results of Gitik and Shelah, where the density character of so simple a space as $(\mathbf{2}^{\aleph_\omega})_{\aleph_1}$ is not determined by the axioms of ZFC (even when $2^{\aleph_n} = \aleph_{n+1}$ for all $n < \omega$ and $2^{\aleph_\omega} = \aleph_{\omega+2}$) as indicating the difficulty, perhaps even the futility, of finding a pleasing and definitive κ-box analogue of Theorem 4.3.

(f) It is clear from the relations

$$|\mathbf{2}^{\aleph_\omega}| \geq d((\mathbf{2}^{\aleph_\omega})_{\aleph_\omega}) \geq d((\mathbf{2}^{\aleph_\omega})_{\aleph_1}) = \aleph_{\omega+2} = |\mathbf{2}^{\aleph_\omega}|$$

in $\mathbb{V}_2$ that $d((\mathbf{2}^{\aleph_\omega})_{\aleph_\omega}) = \aleph_{\omega+2}$ there. For the value of $d((\mathbf{2}^{\aleph_\omega})_{\aleph_\omega})$ in the model $\mathbb{V}_1$ we have

$$\aleph_{\omega+2} = |2^{\aleph_\omega}| \geq d((\mathbf{2}^{\aleph_\omega})_{\aleph_\omega}) \geq d((\mathbf{2}^{\aleph_\omega})_{\aleph_1}) = \aleph_{\omega+1}$$

there, i.e.,

$$\aleph_{\omega+1} \leq d((\mathbf{2}^{\aleph_\omega})_{\aleph_\omega}) \leq \aleph_{\omega+2}.$$

As it was noted in [16, p. 236] there exist models of ZFC such that

$$d((D(\alpha)^{\aleph_\omega})_\kappa) = \aleph_{\omega+1}$$

for every $\alpha, \kappa < \aleph_\omega$ and therefore in such models $d((\mathbf{2}^{\aleph_\omega})_{\aleph_\omega}) = \aleph_{\omega+1}$.

We do not know if there exist models of ZFC such that $d((\mathbf{2}^{\aleph_\omega})_{\aleph_1}) = \aleph_{\omega+1}$ and $d((\mathbf{2}^{\aleph_\omega})_{\aleph_\omega}) = \aleph_{\omega+2}$.

(g) For an exact computation of the weight and Souslin number of the spaces $(\mathbf{2}^{\aleph_\omega})_{\aleph_1}$ and $(\mathbf{2}^{\aleph_\omega})_{\aleph_\omega}$ in the Gitik-Shelah models $\mathbb{V}_1$ and $\mathbb{V}_2$, see Remark 5.29 below.

(h) While we do not pretend to follow every detail of the arguments from [16], nor to frame maximal generalizations, we note that the consistent failure of (4.1) and (4.2) is not restricted to the case $\alpha = 2 < \omega$. In both $\mathbb{V}_1$ and $\mathbb{V}_2$ one evidently has $\aleph_n^{<\aleph_\omega} = \aleph_\omega$ for $0 < n < \omega$, so (4.1) and (4.2) fail in those models (with $\kappa = \aleph_\omega$) for every α such that $2 \leq \alpha < \aleph_\omega$.

(i) In passing we note the existence of two misprints in [16] which have confused at least two readers: Reference in Theorem 1.1(c) should be only to *uncountable* cardinals γ, and in Theorem 4.2(4) the symbol $< \aleph_0$ should be $< \aleph_1$.

Remark 4.15. The arguments developed to prove 4.7–4.13 follow the general pattern of classical arguments used to prove the original Hewitt-Marczewski-Pondiczery Theorem 4.1, albeit with combinatorial modifications necessary to accommodate to the κ-box topology. (When $\kappa = \omega$, Lemma 4.6 reduces to the simple observations that (1) the product of Hausdorff spaces is a Hausdorff space, and (2) in a Hausdorff space, the points of any finite set can be separated by disjoint open sets.) Quite likely, it was reasoning similar to ours which over 40 years ago provoked from Engelking and Karłowicz [7, p. 285], after they had completed their own proof of the Hewitt-Marczewski-Pondiczery theorem, the cavalier statement (here we quote faithfully, but using the notation of the present paper) "We can also derive theorems analogous to those above for κ-box topologies.... We shall not formulate these theorems since they are less interesting, but the reader, if he wishes, will be able to do so without the least difficulty." OK, fair enough. We do note, however, that in the several treatments known to us of the Hewitt-Marczewski-Pondiczery theorem, we have found no mention of the cardinal number $(\alpha^{<\kappa})^{<\kappa}$ which figures prominently and naturally in our development. (This is hardly surprising with respect to the paper [7], since those authors restrict attention to box products of the form $(X_I)_{\kappa^+}$.) Nor have we found an indication, as in Theorem 4.17 below, that the upper bound $\alpha^{<\kappa} \geq d(((D(\alpha))^\beta)_\kappa)$ given in Theorem 4.8(a) is in fact assumed in every case with $\kappa \leq \beta \leq 2^\alpha$.

Part B. Lower Bounds for $d((X_I)_\kappa)$

In Part A, seeking κ-box analogues and generalizations of Theorem 4.1, for specific function pairs f and g of two variables we have sought a function

h so that

$$d(((D(f(\alpha,\kappa)))^{g(\alpha,\kappa)})_\kappa) \leq h(\alpha,\kappa)$$

holds. Now in Part B, again for hand-picked f and g, we seek h' so that

$$d(((D(f(\alpha,\kappa)))^{g(\alpha,\kappa)})_\kappa) \geq h'(\alpha,\kappa). \tag{4.4}$$

In some cases the choice $h = h'$ is accessible, so

$$d(((D(f(\alpha,\kappa)))^{g(\alpha,\kappa)})_\kappa)$$

is computed exactly. In other cases, in parallel with Theorem 4.2, we find several conditions sufficient to ensure that the inequality (4.4) is strict.

Lemma 4.16. *Let $\alpha \geq 2$ and $\kappa \geq \omega$ be cardinals and let $E := (D(\alpha))^\kappa$. Then $d(E_\kappa) \geq \alpha^{<\kappa}$.*

Proof. If the inequality fails there is $\lambda < \kappa$ such that $d(E_\kappa) < \alpha^\lambda$. The space $((D(\alpha))^\lambda)_\kappa$ is then discrete, and since the projection from E_κ onto $((D(\alpha))^\lambda)_\kappa$ is continuous we have the contradiction

$$\alpha^\lambda = d(((D(\alpha))^\lambda)_\kappa) \leq d(E_\kappa) < \alpha^\lambda.$$

□

As we noted in Discussion 4.10, the Hewitt-Marczewski-Pondiczery theorem may be regarded as a routine generalization of this startling special case: $d((D(\alpha))^\beta) = \alpha$ when $\alpha \geq \omega$ and $1 \leq \beta \leq 2^\alpha$. We draw specific attention therefore to the correct κ-box analogue of that result. We note that no regularity hypothesis is imposed here on the cardinal number κ.

Theorem 4.17. *Let $\omega \leq \kappa \leq \beta \leq 2^\alpha$. Then $d((D(\alpha))^\beta)_\kappa) = \alpha^{<\kappa}$.*

Proof. The inequalities $\geq$ and $\leq$ are immediate from Lemma 4.16 and Theorem 4.8(a), respectively. □

In the following theorems we compute the density character of certain specific spaces.

Theorem 4.18. *Let $\omega \leq \kappa$ and $2 \leq \alpha \leq \kappa$. If either $\log(\kappa) < \kappa$ or κ is a regular strong limit cardinal, then*

(a) $2^{<\kappa} = \alpha^{<\kappa} = \kappa^{<\kappa}$; *and*

(b) $d(((D(\alpha))^\kappa)_\kappa) = 2^{<\kappa} = \alpha^{<\kappa} = \kappa^{<\kappa}$.

Proof. (a) That $2^{<\kappa} \leq \alpha^{<\kappa} \leq \kappa^{<\kappa}$ is clear, since $2 \leq \alpha \leq \kappa$. Now if $\log(\kappa) < \kappa$ then

$$\kappa^{<\kappa} \leq (2^{\log(\kappa)})^{<\kappa} = \Sigma_{\lambda<\kappa}\,(2^{\log(\kappa)})^{\lambda} = \Sigma_{\lambda<\kappa}\,2^{\lambda} = 2^{<\kappa};$$

and if κ is regular then since no set in $[\kappa]^{<\kappa}$ is cofinal in κ we have $[\kappa]^{<\kappa} \subseteq \bigcup_{\eta<\kappa} \mathcal{P}(\eta)$, so if in addition κ is a strong limit cardinal then

$$\kappa^{<\kappa} = |[\kappa]^{<\kappa}| \leq \Sigma_{\eta<\kappa}\,|\mathcal{P}(\eta)| = \Sigma_{\eta<\kappa}\,2^{|\eta|} \leq \Sigma_{\eta<\kappa}\,\kappa = \kappa \leq 2^{<\kappa}.$$

(b) From Theorem 4.17 (with $\alpha = \beta = \kappa$ there) and Lemma 4.16 (with $\alpha = 2$ there) we have

$$\kappa^{<\kappa} = d(((D(\kappa))^{\kappa})_{\kappa}) \geq d(((D(\alpha))^{\kappa})_{\kappa}) \geq d((\mathbf{2}^{\kappa})_{\kappa}) \geq 2^{<\kappa},$$

so the asserted equations follow from (a). □

Here is our most comprehensive result for numbers of the form $d((X_I)_{\kappa})$.

Theorem 4.19. *Let $\alpha \geq 2$ and $\kappa \geq \omega$ be cardinals, and let $\alpha \leq \lambda \leq (\alpha^{<\kappa})^{<\kappa}$ and $\kappa \leq \mu \leq 2^{((\alpha^{<\kappa})^{<\kappa})}$. Then*

(a) $\alpha^{<\kappa} \leq d(((D(\lambda))^{\mu})_{\kappa}) \leq (\alpha^{<\kappa})^{<\kappa}$;

(b) *if κ is regular or some $\nu < \kappa$ satisfies $\alpha^{\nu} = \alpha^{<\kappa}$, then* $d(((D(\lambda))^{\mu})_{\kappa}) = \alpha^{<\kappa}$.

Proof. (a) This is clear from Lemma 4.16 and Corollary 4.9(b).

(b) From Theorem 2.8(a) we have $\alpha^{<\kappa} = (\alpha^{<\kappa})^{<\kappa}$, so (b) follows from (a). □

We note next that for $\lambda = \alpha$ and $\kappa \leq \mu \leq 2^{(\alpha^{<\kappa})}$, the conclusion of Theorem 4.19(b) can be established with a supplementary hypothesis weaker than the existence of $\nu < \kappa$ such that $\alpha^{\nu} = \alpha^{<\kappa}$. (The ZFC-consistent existence of instances to which Theorem 4.20 applies, while Theorem 4.19(b) does not, is shown in Remark 4.21.)

Theorem 4.20. *Let $\alpha \geq 2$, $\kappa \geq \omega$ and $\kappa \leq \mu \leq 2^{(\alpha^{<\kappa})}$, and set $E := (D(\alpha))^{\mu}$. If there is $\nu < \kappa$ such that $2^{(\alpha^{\nu})} = 2^{(\alpha^{<\kappa})}$, then $d(E_{\kappa}) = \alpha^{<\kappa}$.*

Proof. That $d(E_{\kappa}) \geq \alpha^{<\kappa}$ is immediate from Lemma 4.16. Now for $\nu \leq \lambda < \kappa$ we have $2^{(\alpha^{\lambda})} = 2^{(\alpha^{<\kappa})}$ and there is, by Theorem 4.8(a) with α, α^{λ}, and λ^{+} in the role of α, β, and κ there, a dense set $A(\lambda) \subseteq ((D(\alpha))^{2^{(\alpha^{\lambda})}})_{\lambda^{+}}$ such that $|A(\lambda)| \leq (\alpha \cdot \alpha^{\lambda})^{\lambda} = \alpha^{\lambda}$. The set $A := \bigcup_{\nu\leq\lambda<\kappa} A(\lambda)$ is clearly dense in $(((D(\alpha))^{(2^{(\alpha^{<\kappa})})})_{\kappa}$, so $d(E_{\kappa}) \leq d((((D(\alpha))^{(2^{(\alpha^{<\kappa})})})_{\kappa}) \leq |A| \leq \Sigma_{\nu\leq\lambda<\kappa}\,\alpha^{\lambda} \leq \kappa \cdot \alpha^{<\kappa} = \alpha^{<\kappa}$. □

Remarks 4.21. (a) We indicate that there are models $\mathbb{M}$ of ZFC in which, for suitably chosen α and κ as in Theorem 4.20 (specifically for $\alpha = 2$, $\kappa = \aleph_\omega$) there exist $\nu < \kappa$ such that $2^{\alpha^\nu} = 2^{(\alpha^{<\kappa})}$ but there is no $\nu < \kappa$ such that $\alpha^\nu = \alpha^{<\kappa}$. To that end, using the fundamental consistency theorem of Easton [6] as exposed by Kunen [22, VIII], let $\mathbb{M}$ be a model of ZFC in which

(1) $2^{\aleph_n} = \aleph_{\omega+n+1}$ for $n < \omega$,

(2) $2^{\aleph_\omega} = \aleph_{\omega+\omega+1}$, and

(3) $2^{\aleph_{\omega+n+1}} = 2^{\aleph_{\omega+\omega}} = \aleph_{\omega+\omega+2}$ for $n < \omega$.

It is clear in $\mathbb{M}$, taking $\alpha = 2$ and $\kappa = \aleph_\omega$, that

$$\alpha^{<\kappa} = 2^{<\aleph_\omega} = \aleph_{\omega+\omega},$$

so for every $\nu = \aleph_n < \aleph_\omega = \kappa$ we have

$$\alpha^\nu = 2^{\aleph_n} = \aleph_{\omega+n+1} < \aleph_{\omega+\omega} = \alpha^{<\kappa}$$

and

$$2^{\alpha^\nu} = 2^{(2^{\aleph_n})} = \aleph_{\omega+\omega+2} = 2^{\aleph_{\omega+\omega}} = 2^{(\alpha^{<\kappa})}.$$

(b) We note in passing that the existence of $\nu < \kappa$ such that $2^{\alpha^\nu} = 2^{(\alpha^{<\kappa})}$ holds in all models in which $\log(2^{(\alpha^{<\kappa})}) < \alpha^{<\kappa}$. Indeed if $\log(2^{(\alpha^{<\kappa})}) = \beta < \alpha^{<\kappa}$ then there is $\lambda < \kappa$ such that $\beta < \alpha^\lambda$, and then $2^\beta = 2^{\alpha^\nu} = 2^{(\alpha^{<\kappa})}$ for all ν satisfying $\lambda \le \nu < \kappa$.

Next as promised we give a couple of generalizations of Theorem 4.2 to the κ-box context.

Theorem 4.22. *Let $\alpha \ge \omega$, $\beta \ge 2$, $\kappa \ge \omega$ and let $S(X_i) > \beta$ for each $i \in I$. Suppose that either*

(i) *some $i \in I$ satisfies $d(X_i) > \alpha$; or*

(ii) *$|[I]^{<\kappa}| > 2^\alpha$; or*

(iii) *$\beta^{<\kappa} > 2^\alpha$; or*

(iv) *there is $J \in [I]^{<\kappa}$ such that $\beta^{|J|} > \alpha$.*

Then $d((X_I)_\kappa) > \alpha$.

Proof. The sufficiency of (i) is clear: the natural projection from $(X_I)_\kappa$ to X_i is continuous and surjective, so $d((X_I)_\kappa) \ge d(X_i)$.

For the rest of the proof, for $i \in I$ let $\{U_i(\eta) : \eta < \beta\}$ be a cellular family in X_i.

We prove $d((X_I)_\kappa) > \alpha$, assuming that either (ii) or (iii) holds. For $A \in [I]^{<\kappa}$ and $f \in \beta^A$ set

$$U(A,f) := \{x \in X_I : i \in A \Rightarrow x_i \in U_i(f(i))\}.$$

Let T be dense in $(X_I)_\kappa$ with $|T| = d((X_I)_\kappa)$ and set $T(A,f) := T \cap U(A,f)$.

We claim that the map $\phi : \bigcup_{A\in[I]^{<\kappa}} (A \times \beta^A) \longrightarrow \mathcal{P}(T)$ given by $\phi(A,f) = T(A,f)$ is injective. Let $(A,f) \neq (B,g)$ with $A, B \in [I]^{<\kappa}$, $f \in \beta^A$, and $g \in \beta^B$. We consider two cases.

Case 1. $A = B$. Then there is $i \in A = B$ such that $f(i) \neq g(i)$, so $T(A,f) \cap T(B,g) = \emptyset$ (since $U_i(f(i)) \cap U_i(g(i)) = \emptyset$).

Case 2. $A \neq B$. Without loss of generality there is then $i \in A \backslash B$. Choose $\eta < \beta$ such that $f(i) \neq \eta$. The set $V := U(B,g) \cap \pi_i^{-1}(U_i(\eta))$ is then nonempty and open in $(X_I)_\kappa$, and with $p \in T \cap V$ we have $p \in T(B,g) \backslash T(A,f)$.

The claim is proved.

For $\lambda < \kappa$ and $A \in [I]^\lambda$ we have $A \times \beta^A \subseteq \mathrm{dom}(\phi)$, so $|\mathrm{dom}(\phi)| \geq |[I]^\lambda|$ and $|\mathrm{dom}(\phi)| \geq \beta^\lambda$. Then it follows that $|\mathrm{dom}(\phi)| \geq |[I]^{<\kappa}| \cdot \beta^{<\kappa}$, so if $d((X_I)_\kappa) = |T| \leq \alpha$ and (ii) or (iii) holds we would have the contradiction

$$2^\alpha < |[I]^{<\kappa}| \cdot \beta^{<\kappa} \leq |\mathrm{dom}(\phi)| \leq |\mathcal{P}(T)| \leq 2^\alpha.$$

It remains to derive $d((X_I)_\kappa) > \alpha$ from (iv). Let J be as hypothesized, and for $f \in \beta^J$ set

$$V(f) := (\Pi_{i\in J}\, U_i(f(i))) \times (\Pi_{i\in I\backslash J}\, X_i).$$

Then $\mathcal{V} := \{V(f) : f \in \beta^J\}$ is cellular in $(X_I)_\kappa$, so

$$d((X_I)_\kappa) \geq |\mathcal{V}| = |\beta^J| = \beta^{|J|} > \beta.$$

□

We note that the hypothesis in Theorem 4.22 on the family $\{X_i : I \in I\}$ can be relaxed in places. In connection with (iv), for example, it is clear that the condition $S(X_i) > \beta$ need hold only for i in some set $J \in [I]^{<\kappa}$ such that $\beta^{|J|} > \alpha$.

Taking $\beta = 2$ in Theorem 4.22 and replacing α there first by $\alpha^{<\kappa}$ and then by $(\alpha^{<\kappa})^{<\kappa}$, we obtain respectively parts (a) and (b) of the following corollary.

Corollary 4.23. *Let $\alpha \geq 2$ and $\kappa \geq \omega$ be cardinals, and let $\{X_i : i \in I\}$ be a set of spaces such that $S(X_i) \geq 3$ for each $i \in I$.*

(a) *If $|[I]^{<\kappa}| > 2^{(\alpha^{<\kappa})}$ then $d((X_I)_\kappa)) > \alpha^{<\kappa}$; and*

(b) *if $|[I]^{<\kappa}| > 2^{((\alpha^{<\kappa})^{<\kappa})}$ then $d((X_I)_\kappa) > (\alpha^{<\kappa})^{<\kappa}$.*

Corollary 4.23 shows that the inequalities given in Corollary 4.11 are sharp. The following simple combinatorial result offers reformulations of some of the hypotheses of Corollary 4.23.

Theorem 4.24. *Let $\alpha \geq 2$ and $\kappa \geq \omega$ be cardinals and let I be a set.*

(a) *These three conditions are equivalent:*

(1) $|I| > 2^{(\alpha^{<\kappa})}$; (2) $|[I]^{<\kappa}| > 2^{(\alpha^{<\kappa})}$; (3) $|I|^{<\kappa} > 2^{(\alpha^{<\kappa})}$.

(b) *These three conditions are equivalent.*

(1) $|I| > 2^{((\alpha^{<\kappa})^{<\kappa})}$; (2) $|[I]^{<\kappa}| > 2^{((\alpha^{<\kappa})^{<\kappa})}$; (3) $|I|^{<\kappa} > 2^{((\alpha^{<\kappa})^{<\kappa})}$.

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are clear in both (a) and (b). To see that (3) $\Rightarrow$ (1) in (a), note that if $|I| \leq 2^{(\alpha^{<\kappa})}$ then

$$|I|^{<\kappa} \leq (2^{(\alpha^{<\kappa})})^{<\kappa} \leq (2^{(\alpha^{<\kappa})})^{\kappa} = 2^{(\alpha^{<\kappa})\cdot\kappa} = 2^{(\alpha^{<\kappa})}.$$

The proof that (3) $\Rightarrow$ (1) in (b) is similar. □

Remarks 4.25. (a) The authors of [3, 3.16], improving their results from [2], show that if $E = (D(\alpha))^{2^\alpha}$ or $E = (D(\alpha))^{\alpha^+}$, then $d(E_\kappa) = \alpha$ if and only if $\alpha = \alpha^{<\kappa}$. Clearly Theorem 4.19(b) above improves that statement. Similarly, Corollary 4.11(a) improves [3, 3.18], which asserts the conclusion of 4.11 only under the assumption that $\alpha = \alpha^{<\kappa}$.

(b) The investigation by Hu [18] of cardinals of the form $d((X_I)_\kappa)$ is from a different perspective: Rather than beginning with the set $E = \Pi_{i\in I} D(\alpha_i)$ and seeking dense subsets of the space E_κ, Hu [18] uses (maximal) generalized independent families of partitions of a given set S to map S faithfully onto dense subsets of spaces of the form E_κ (one writes $S \subseteq E_\kappa$). The emphasis is on finding conditions so that $S \subseteq E_\kappa$ is irresolvable. Hu [18] shows, for example, that if each α_i is less than the first cardinal which is strongly κ-inaccessible, and E_κ contains a dense, irresolvable subspace, then $\kappa = 2^{<\kappa}$, and consistently a measurable cardinal exists.

5 On the Souslin number of κ-box products

We remind the reader of our standing convention that hypothesized spaces are not assumed to enjoy any special separation properties. This complicates our exposition slightly, since it is convenient for us to cite some basic familiar results from sources where, for simplicity and often unnecessarily, such properties as Hausdorff separation are assumed throughout. We mention in particular the following two useful results, both valid for every space. These will be used frequently in what follows, without explicit restatement.

Let X be a space. Then

(a) $S(X) \neq \omega$*; and*

(b) *either* $S(X) < \omega$ *or* $S(X)$ *is an (infinite) regular cardinal.*

The proof given in [4, 2.10] of (a), although long-winded and unnecessarily complicated, is valid without separation assumptions; (b) is a fundamental result of Erdős and Tarski [13] (see [3, 2.10], [4, 2.14] for other treatments).

As with Sections 2 and 3 concerning weight and density character respectively, we begin this section by citing those classical Souslin-related theorems (pertaining to the usual product topology) whose κ-box analogues we study here. As usual, when a set $\{X_i : i \in I\}$ of spaces is given we write $X_I := \Pi_{i \in I} X_i$.

Theorem 5.1. *Let* $\{X_i : i \in I\}$ *be a set of nonempty spaces and set*

$$\alpha := \sup\{S(X_F) : \emptyset \neq F \in [I]^{<\omega}\}.$$

Then

$$S(X_I) = \left\{ \begin{array}{cl} \alpha & \text{if (a) } \alpha < \omega \text{ or (b) } \alpha \text{ is regular and } \ \alpha > \omega \\ \alpha^+ & \text{in all other cases} \end{array} \right\}.$$

The thrust of Theorem 5.1 is that the Souslin number of a product space X_I (in the usual product topology) is completely determined by the Souslin numbers of the various subproducts X_F with $F \in [I]^{<\omega}$. Much of this Section is devoted to the presentation of κ-box analogues of Theorem 5.1. See in particular Theorems 5.7, 5.8, 5.23, 5.48, and Corollaries 5.43(b), 5.49.

The proof of Theorem 5.1 depends on nontrivial combinatorial machinery in which, reflecting the restriction to the usual product topology, the cardinal numbers ω and ω^+ figure prominently. The key to the proof is the theory of quasi-disjoint families as developed by Erdős and Rado [11], [12] (the "Δ-system lemma"); this is used in the proof of Theorem 5.7. For a thorough development of that result and of several other Souslin-related consequences, the reader may consult [3, 3.8] and [4, 3.25].

As we noted in Theorem 4.1, for $\alpha \geq \omega$ the product of 2^α-many (or fewer) spaces X_i, with each $d(X_i) \leq \alpha$, satisfies $d(X_I) \leq \alpha$. From that and Theorem 5.1, one can derive this well-known theorem (see for example [8, 2.3.17]; or see Theorem 5.14 for the general κ-box statement of which Theorem 5.2 is the case $\kappa = \omega$).

Theorem 5.2. *Let* $\alpha \geq \omega$ *and* $\{X_i : i \in I\}$ *be a set of spaces with* $d(X_i) \leq \alpha$ *for each* $i \in I$*. Then* $S(X_I) \leq \alpha^+$.

Discussion 5.3. Theorems 5.1 and 5.2 leave unanswered even for the usual product topology a question which arises naturally in their wake:

Given $\alpha \geq \omega$ and a finite set $\{X_i : i \in F\}$ of spaces with each $S(X_i) \leq \alpha$, is necessarily $S(X_F) \leq \alpha$?

The brief response is that the question is not settled by the axioms of ZFC, even in the case $\alpha = \omega^+$. Referring the reader to [4] for extensive comments and relevant bibliographic citations, we remark simply that it has been known in ZFC for many years that while the Souslin number may "jump" in passing from a space X to $X \times X$, roughly speaking that jump is bounded by a single exponential. To be more precise we give below a theorem taken from [3, 3.13] that gives one possible generalization of that claim for the κ-box topology. For its full generalization to the κ-box context, see Theorem 5.33.

Theorem 5.4. *If $\alpha \geq \omega$ and $\{X_i : i \in I\}$ is a family of spaces such that $S(X_i) \leq \alpha^+$ for $i \in I$, then $S((X_I)_{\alpha^+}) \leq (2^\alpha)^+$.*

The following notational device (see [4, p. 254]) is useful as we seek κ-box analogues of Theorems 5.1 and 5.2.

Notation 5.5. Let α and κ be infinite cardinals. Then α is *strongly κ-inaccessible* (in symbols: $\kappa \ll \alpha$) if (a) $\kappa < \alpha$ and (b) $\beta^\lambda < \alpha$ whenever $\beta < \alpha$ and $\lambda < \kappa$.

Remark 5.6. To help the reader fix ideas, we note that the condition $\kappa \ll \alpha$ occurs for many pairs of cardinals. For example,

(1) every uncountable cardinal α satisfies $\omega \ll \alpha$;

(2) every infinite cardinal α satisfies $\alpha^+ \ll (2^\alpha)^+$, since if $\lambda < \alpha^+$ and $\beta < (2^\alpha)^+$ then $\lambda \leq \alpha$ and $\beta \leq 2^\alpha$ and hence $\beta^\lambda \leq (2^\alpha)^\alpha = 2^\alpha < (2^\alpha)^+$; and

(3) every pair κ, α with $\alpha \geq 2$ and $\kappa \geq \omega$ satisfies $\kappa \ll ((\alpha^{<\kappa})^{<\kappa})^+$, since if $\lambda < \kappa$ and $\beta < ((\alpha^{<\kappa})^{<\kappa})^+$ then $\beta \leq (\alpha^{<\kappa})^{<\kappa}$ and hence $\beta^\lambda \leq ((\alpha^{<\kappa})^{<\kappa})^{<\kappa} = (\alpha^{<\kappa})^{<\kappa}$ (by Theorem 2.6(d)).

(4) every pair κ, α with $\alpha \geq 2$ and $\kappa \geq \omega$ singular satisfies $\kappa^+ \ll ((\alpha^{<\kappa})^{<\kappa})^+$, since if $\lambda < \kappa^+$ and $\beta < ((\alpha^{<\kappa})^{<\kappa})^+$ then $\lambda \leq \kappa$ and $\beta \leq (\alpha^{<\kappa})^{<\kappa}$ and hence

$$\beta^\lambda \leq ((\alpha^{<\kappa})^{<\kappa})^\kappa = \alpha^\kappa = (\alpha^{<\kappa})^{<\kappa}$$

(by Theorem 2.6(c)).

Theorem 5.7 (cf. [3, 3.8], [4, 3.25(a)])**.** *Let $\omega \leq \kappa \ll \alpha$ with α regular and let $\{X_i : i \in I\}$ be a family of nonempty spaces. Then $S((X_I)_\kappa) \leq \alpha$ if and only if $S((X_J)_\kappa) \leq \alpha$ for each nonempty $J \in [I]^{<\kappa}$.*

From the relations $\alpha^+ \ll (2^\alpha)^+$ and $\kappa \ll ((\alpha^{<\kappa})^{<\kappa})^+$ we have these consequences of Theorem 5.7.

Theorem 5.8. *Let $\kappa \geq \omega$ and $\alpha \geq 2$ be cardinals and let $\{X_i : i \in I\}$ be a family of nonempty spaces. Then*

(a) *If $\alpha \geq \omega$, then $S((X_I)_{\alpha^+}) \leq (2^\alpha)^+$ if and only if*

$$S((X_J)_{\alpha^+}) \leq (2^\alpha)^+$$

for each nonempty $J \in [I]^{\leq\alpha}$; and

(b) *$S((X_I)_\kappa) \leq ((\alpha^{<\kappa})^{<\kappa})^+$ if and only if*

$$S((X_J)_\kappa) \leq ((\alpha^{<\kappa})^{<\kappa})^+$$

for each nonempty $J \in [I]^{<\kappa}$.

The following result, another immediate consequence of Theorem 5.7, furnishes in certain cases an exact formula for the numbers $S((X_I)_\kappa)$.

Corollary 5.9. *Let $\kappa \geq \omega$, let $\{X_i : i \in I\}$ be a set of nonempty spaces, and set $\alpha := \sup\{S((X_J)_\kappa) : \emptyset \neq J \in [I]^{<\kappa}\}$. Then*

(a) (cf. [4, 3.27]) *If α is regular and $\kappa \ll \alpha$ then $S((X_I)_\kappa) = \alpha$;*

(b) *if α is singular and $\kappa \ll \alpha^+$, then $S((X_I)_\kappa) = \alpha^+$.*

The following result, taken from [4, 3.28], is given here for the reader's convenience. Since every infinite cardinal α satisfies $\omega \ll \alpha^+$ with α^+ regular, the implication (a) ⇒ (b) is a suitable κ-box analogue of Theorem 5.2.

Theorem 5.10. *Let $\omega \leq \kappa < \alpha$ with α regular. Then these conditions are equivalent.*

(a) *$\kappa \ll \alpha$;*

(b) *if $\{X_i : i \in I\}$ is a set of spaces with each $d(X_i) < \alpha$, then $S((X_I)_\kappa) \leq \alpha$.*

Proof. (a) ⇒ (b). According to Theorem 5.7, it suffices to show that $S((X_J)_\kappa) \leq \alpha$ whenever $\emptyset \neq J \in [I]^{<\kappa}$. Fix such J, for $i \in J$ let D_i be dense in X_i with $|D_i| = \beta_i < \alpha$, and set $D := \Pi_{i\in J} D_i$ and $\beta := \sup_{i\in J} \beta_i$. Since $|J| < \kappa < \alpha = \mathrm{cf}(\alpha)$ we have $\beta < \alpha$, and from $\kappa \ll \alpha$ follows $|D| \leq \beta^{|J|} < \alpha$. Clearly D is dense in $(X_J)_\kappa$, and from $d((X_J)_\kappa) < \alpha$ it then follows that $S((X_J)_\kappa) \leq \alpha$, as required.

(b) ⇒ (a). Fix $\beta < \alpha$ and $\lambda < \kappa$, and set $X := (D(\beta))^\lambda$. Then $(X)_\kappa$ is discrete, and from $S((X)_\kappa) \leq \alpha$ it follows that $\beta^\lambda = |X| = |(X)_\kappa| < \alpha$. □

Corollary 5.11. *Let $\alpha \geq 2$ and $\kappa \geq \omega$, and let $\{X_i : i \in I\}$ be a set of spaces.*

(a) *If $\alpha \geq \omega$ and $d(X_i) \leq 2^\alpha$ for each $i \in I$, then $S((X_I)_{\alpha^+}) \leq (2^\alpha)^+$; and*

(b) *if $d(X_i) \leq (\alpha^{<\kappa})^{<\kappa}$ for each $i \in I$, then $S((X_I)_\kappa) \leq ((\alpha^{<\kappa})^{<\kappa})^+$.*

Proof. As noted in Remark 5.6(3) and 5.6(4) we have $\alpha^+ \ll (2^\alpha)^+$ and $\kappa \ll (\alpha^{<\kappa})^{<\kappa})^+$, so Theorem 5.10 applies (with $(2^\alpha)^+$ replacing α in (a) and with $((\alpha^{<\kappa})^{<\kappa})^+$ replacing α in (b)). □

The following result, which we are going to use frequently, shows a relationship between the Souslin number of product spaces of the type $(X_I)_\kappa$ and the cardinal number κ.

Lemma 5.12. *Let $\alpha \geq 3$ and $\kappa \geq \omega$ be cardinals, and let $\{X_i : i \in I\}$ be a set of spaces such that $S(X_i) \geq \alpha$ for each $i \in I$. Let $\mu = \min\{\kappa, |I|^+\}$. Then $S((X_I)_\kappa) > \beta^{<\mu}$ for each $\beta < \alpha$.*

Proof. We show first that if $\kappa \geq |I|$ then $S((X_I)_\kappa) > \kappa$. Indeed, in this case there is $J \in [I]^\kappa$, say $J = \{i_\eta : \eta < \kappa\}$. For $i_\eta \in J$ let $U(i_\eta, 0)$ and $U(i_\eta, 1)$ be nonempty, disjoint open subsets of X_{i_η} and for $\eta < \kappa$ set

$$U(\eta) := (\Pi_{\xi<\eta} U(i_\xi, 0) \times (U(i_\eta, 1)) \times (\Pi_{i \in I \setminus \{i_\xi : \xi \leq \eta\}} X_i).$$

Then $\mathcal{C}(\kappa) := \{U(\eta) : \eta < \kappa\}$ is cellular in $(X_I)_\kappa$, so $S((X_I)_\kappa) > |\mathcal{C}(\kappa)| = \kappa$.

Now fix $\beta < \alpha$, $\lambda < \mu$ and $J \in [I]^\lambda$. For $i \in J$ let $\{U(i, \eta) : \eta < \beta\}$ be a cellular family in X_i, and for $f \in \beta^J$ set $U(f) := (\Pi_{i \in J} U(i, f(i))) \times (X_{I \setminus J})$. Then $\mathcal{C} := \{U(f) : f \in \beta^J\}$ is cellular in $(X_I)_\kappa$, and

$$S((X_I)_\kappa) > |\mathcal{C}| = |\beta^J| = \beta^\lambda. \tag{5.1}$$

Since $S((X_I)_\kappa) < \beta^\lambda$ for each $\lambda < \kappa$, we have

$$S((X_I)_\kappa) \geq \beta^{<\mu} \text{ for each } \beta < \alpha.$$

To show that $S((X_I)_\kappa) > \beta^{<\mu}$ for each $\beta < \alpha$ we consider three cases.

Case 1. The cardinal number $\beta^{<\mu}$ is singular. Then clearly $S((X_I)_\kappa) > \beta^{<\mu}$.

Case 2. The cardinal number $\beta^{<\mu}$ is regular and there is $\nu < \mu$ such that $\beta^\nu = \beta^{<\mu}$. Then $S((X_I)_\kappa) > \beta^\nu = \beta^{<\mu}$ by (5.1).

Case 3. Cases 1 and 2 fail. Then, according to Lemma 5.18(b), $\beta^{<\mu} = \mu$ and μ is a regular strong limit cardinal. Since $\mu = \min\{\kappa, |I|^+\}$, we have $\mu = \kappa$, hence $|I| \geq \kappa$ and since in that case $S((X_I)_\kappa) > \kappa$ we conclude that $S((X_I)_\kappa) > \beta^{<\mu}$. □

Theorem 5.13. *Let $\kappa \geq \omega$ be a limit cardinal and let $\{X_i : i \in I\}$ be a set of spaces such that $|I| \geq \kappa$ and $S(X_i) \geq 3$ for each $i \in I$. Let also*

$$\alpha := \sup\{S((X_I)_\gamma) : \gamma < \kappa\} \text{ for } \kappa > \omega \text{ and}$$

$$\alpha := \sup\{S(X_J) : J \in [I]^{<\kappa}\} \text{ for } \kappa = \omega.$$

Then

(a) $\kappa \leq \alpha \leq S((X_I)_\kappa) \leq \alpha^+$, *and* $\kappa^+ \leq S((X_I)_\kappa)$;

(b) *if α is regular and $\kappa < \alpha$ then $S((X_I)_\kappa) = \alpha$; and*

(c) *if α is singular or $\kappa = \alpha$ then $S((X_I)_\kappa) = \alpha^+$.*

Proof. In each of (a), (b), or (c) the case $\kappa = \omega$ follows from Theorem 5.1. Therefore below we consider only the case $\kappa > \omega$.

(a) That $\alpha \leq S((X_I)_\kappa)$ is obvious. Let $\mu = \min\{\kappa, |I|^+\}$. Since $|I| \geq \kappa$, we have $\mu = \kappa$. Then it follows from Lemma 5.12 that $S((X_I)_\kappa) > 2^{<\mu} = 2^{<\kappa} \geq \kappa$, hence $S((X_I)_\kappa) \geq \kappa^+$. Also, since $\kappa > \omega$, $S((X_I)_{\gamma^+}) > 2^{<\gamma^+} = 2^\gamma \geq \gamma^+$ for every $\gamma < \kappa$, hence $\alpha \geq \kappa$.

To prove $S((X_I)_\kappa) \leq \alpha^+$, suppose there is in $(X_I)_\kappa$ a basic cellular family $\mathcal{C}$ such that $|\mathcal{C}| = \alpha^+$, and for $\gamma < \kappa$ set $C(\gamma) := \{U \in \mathcal{C} : |R(U)| < \gamma\}$. Then since α^+ is regular with $\alpha^+ > \kappa$, there is $\gamma < \kappa$ such that $|\mathcal{C}(\gamma)| = \alpha^+$ and we have the contradiction $\alpha \geq S((X_I)_\gamma) \geq \alpha^{++}$.

(b) A similar argument applies. If in $(X_I)_\kappa$ there is a basic cellular family $\mathcal{C}$ such that $|\mathcal{C}| = \alpha$ then with

$$\mathcal{C}(\gamma) := \{U \in \mathcal{C} : |R(U)| < \gamma\} \text{ for } \gamma < \kappa$$

we have $\mathcal{C} = \bigcup_{\gamma<\kappa} \mathcal{C}(\gamma)$ and from the regularity of α and the relation $\kappa < \alpha$ we have $|\mathcal{C}(\gamma)| = \alpha$ for some $\gamma < \kappa$ and then

$$\alpha \geq S((X_I))_\kappa) \geq S((X_I)_\gamma) > \alpha,$$

a contradiction.

(c) Since $S((X_I)_\kappa)$ is regular, this is immediate from (a). □

Theorem 5.14. *Let $\alpha \geq 2$ and $\kappa \geq \omega$ be cardinals and let $\{X_i : i \in I\}$ be a set of spaces with $d(X_i) \leq \alpha$ for each $i \in I$. Then $S((X_I)_\kappa) \leq (\alpha^{<\kappa})^+$.*

Proof. We assume first that $\alpha \geq \omega$ and we consider two cases.

Case 1. $\alpha^{<\kappa} = (\alpha^{<\kappa})^{<\kappa}$. The conclusion is immediate from Corollary 5.11 (even with the hypothesis $d(X_i) \leq \alpha$ weakened to $d(X_i) \leq \alpha^{<\kappa} = (\alpha^{<\kappa})^{<\kappa}$).

<u>Case 2</u>. Case 1 fails. Then κ is singular (by Theorem 2.8(a)) and therefore a limit cardinal such that $\kappa > \omega$. If there exists $\gamma < \kappa$ such that $S((X_I)_\kappa) = S((X_I)_{\gamma^+})$ then since γ^+ is regular it follows from Corollary 5.11(b) that

$$S((X_I)_\kappa) = S((X_I)_{\gamma^+}) \leq (\alpha^\gamma)^\gamma \leq \alpha^{<\kappa} < (\alpha^{<\kappa})^+.$$

If there is no $\gamma < \kappa$ such that $S((X_I)_\kappa) = S((X_I)_{\gamma^+})$ then for each $\gamma < \kappa$ we have $S((X_I)_{\gamma^+}) \leq (\alpha^\gamma)^\gamma \leq \alpha^{<\kappa}$ and hence

$$S((X_I)_\kappa) \leq (\sup_{\gamma<\kappa} S((X_I)_\gamma))^+ \leq (\alpha^{<\kappa})^+$$

from Theorem 5.13(a).

It remains to consider the case $\alpha < \omega$. Note that $d(X_i) \leq \omega$ for each $i \in I$. Then if $\kappa = \omega$ we have

$$S((X_I)_\kappa) = S(X_I) \leq \omega^+ = (\alpha^{<\kappa})^+$$

from Theorem 5.2, and if $\kappa > \omega$ then the preceding paragraphs apply to give

$$S((X_I)_\kappa) \leq (\omega^{<\kappa})^+ \leq ((2^\omega)^{<\kappa})^+ = (2^{<\kappa})^+ \leq (\alpha^{<\kappa})^+.$$

□

Remark 5.15. (a) If the hypothesis $d(X_i) \leq \alpha$ of Theorem 5.14 is weakened to $d(X_i) \leq \alpha^{<\kappa}$, the conclusion can fail. To see that, it is enough to refer to Discussion 4.14, where we noted that for every pre-assigned $\alpha \geq \omega$ the choice $\kappa := \beth_\lambda(\alpha)$ with $\lambda \leq \mathrm{cf}(\alpha)$ guarantees that the space $E := (D(\alpha^{<\kappa}))^I$ with $|I| = \alpha$ has E_κ discrete (since $\kappa > |I|$) and $|E_\kappa| = (\alpha^{<\kappa})^\alpha = \kappa^\alpha > \alpha^{<\kappa}$, hence $S(E_\kappa) = |E_\kappa|^+ > (\alpha^{<\kappa})^+$.

(b) With Theorem 5.14 in hand the implication (a)⇒(b) in Theorem 5.10 becomes now a direct corollary. Indeed, if $\alpha = \beta^+$ in Theorem 5.10 then $d(X_i) \leq \beta$ and according to Theorem 5.14 we have

$$S((X_I)_\kappa) \leq (\beta^{<\kappa})^+ = (\Sigma_{\lambda<\kappa}\, \beta^\lambda)^+ \leq (\beta \cdot \kappa)^+ = \alpha$$

since $\kappa \ll \alpha$. And if α is a regular limit cardinal in Theorem 5.10 then Theorem 5.14 gives $S((X_I)_\kappa) \leq (\alpha^{<\kappa})^+$. But in this case, since α is regular and no set in $[\alpha]^{<\kappa}$ is cofinal in α we have

$$\alpha^{<\kappa} = |[\alpha]^{<\kappa}| \leq \Sigma_{\eta<\alpha}\, |[\eta]^{<\kappa}| = \Sigma_{\eta<\alpha}\, |\eta|^{<\kappa} \leq \Sigma_{\eta<\alpha}\, \alpha = \alpha,$$

since $|\eta|^{<\kappa} = \Sigma_{\zeta<\kappa}\, |\eta|^\zeta \leq \kappa \cdot \alpha = \alpha$ for every $\eta < \alpha$ whenever $\kappa \ll \alpha$.

Theorem 5.17, using some of those same ideas, strengthens that result. For use in its proof and frequently thereafter we adopt henceforth the following notational convention concerning limit cardinals κ. We do not exclude here the possibility that κ is regular, but this convention will be invoked chiefly in cases where it is known that $\mathrm{cf}(\kappa) < \kappa$.

Notation 5.16. Let $\kappa \geq \omega$ be a limit cardinal. Then $\{\kappa_\eta : \eta < \mathrm{cf}(\kappa)\}$ is a set of cardinals such that

(a) $\kappa_\eta < \kappa_{\eta'} < \kappa$ when $\eta < \eta' < \mathrm{cf}(\kappa)$, and

(b) $\Sigma_{\eta<\mathrm{cf}(\kappa)}\, \kappa_\eta = \kappa$.

Theorem 5.17. *Let $\alpha \geq 2$, let $\kappa > \omega$ be a (possibly regular) limit cardinal, and let $\{\kappa_\eta : \eta < \mathrm{cf}(\kappa)\}$ be a family of cardinals as in Notation 5.16. For $\eta < \mathrm{cf}(\kappa)$ let $\{X(\eta) : \eta < \mathrm{cf}(\kappa)\}$ be a (not necessarily faithfully indexed) set of spaces such that $S(X(\eta)) \geq \alpha^{\kappa_\eta}$ for each $\eta < \mathrm{cf}(\kappa)$, and let $X := \Pi_{\eta<\mathrm{cf}(\kappa)}\, X(\eta)$. Then*

(a) $S(X_{\mathrm{cf}(\kappa)}) \geq \alpha^{<\kappa}$; and

(b) if $\alpha^{<\kappa} < (\alpha^{<\kappa})^{<\kappa}$, then $S(X_{(\mathrm{cf}(\kappa))^+}) > \alpha^\kappa \geq (\alpha^{<\kappa})^+$.

Proof. (a) is obvious, since $S(X_{\mathrm{cf}(\kappa)}) \geq S(X(\eta)) \geq \alpha^{\kappa_\eta}$ for each $\eta < \mathrm{cf}(\kappa)$.

(b) The topology of $X_{(\mathrm{cf}(\kappa))^+}$ is the (full) box topology. Since $\mathrm{cf}(\alpha^{<\kappa}) = \mathrm{cf}(\kappa) < \kappa$ by Theorem 2.8, we may assume without loss of generality that $\alpha^{\kappa_\eta} < \alpha^{\kappa_{\eta'}}$ for $\eta < \eta' < \mathrm{cf}(\kappa)$. Let $\mathcal{C}(\eta) := \{X(\eta)\}$ for limit ordinals $\eta < \mathrm{cf}(\kappa)$, and for $\eta < \mathrm{cf}(\kappa)$ let $\mathcal{C}(\eta+1)$ be cellular in $X(\eta+1)$ with $|\mathcal{C}(\eta+1)| \geq \alpha^{\kappa_\eta}$. Then $\mathcal{C} := \{\Pi_{\eta<\mathrm{cf}(\kappa)}\, C_\eta : C_\eta \in \mathcal{C}(\eta)\}$ is cellular in $X_{(\mathrm{cf}(\kappa))^+}$, with

$$|\mathcal{C}| = \Pi_{\eta<\mathrm{cf}(\kappa)}\, |\mathcal{C}(\eta)| \geq \Pi_{\eta<\mathrm{cf}(\kappa)}\, \alpha^{\kappa_\eta} = \alpha^{\Sigma_{\eta<\mathrm{cf}(\kappa)}\, \kappa_\eta} = \alpha^\kappa,$$

so $S(X_{(\mathrm{cf}(\kappa))^+}) > \alpha^\kappa \geq (\alpha^{<\kappa})^+$. □

The following simple lemma, strictly set-theoretic (non-topological) in nature, is one of several preliminaries required for the proof of Theorem 5.23.

Lemma 5.18. *Let $\alpha \geq 2$ and $\kappa \geq \omega$ be cardinals.*

(a) If $\alpha^{<\kappa}$ is a successor cardinal then there is $\nu < \kappa$ such that $\alpha^\nu = \alpha^{<\kappa}$.

(b) If $\alpha^{<\kappa}$ is a regular cardinal and there is no $\nu < \kappa$ such that $\alpha^\nu = \alpha^{<\kappa}$ then $\alpha^{<\kappa} = \kappa$ and κ is a regular strong limit cardinal.

Proof. (a) Let $\alpha^{<\kappa} = \lambda^+$.

If $\kappa = \alpha^{<\kappa}$ then $\alpha^{<\kappa} = \alpha^\lambda$ (with $\lambda < \kappa$).

If $\kappa < \alpha^{<\kappa}$ and $\alpha^\nu < \alpha^{<\kappa}$ for each $\nu < \kappa$, then we have the contradiction

$$\alpha^{<\kappa} = \Sigma_{\nu<\kappa}\, \alpha^\nu \leq \lambda \cdot \kappa = \lambda < \lambda^+ = \alpha^{<\kappa}.$$

(b) It follows from (a) that if (b) fails and there is no $\nu < \kappa$ such that $\alpha^\nu = \alpha^{<\kappa}$, then $\alpha^{<\kappa}$ is a (regular) limit cardinal and we have

$$\alpha^{<\kappa} = \mathrm{cf}(\alpha^{<\kappa}) \leq \mathrm{cf}(\kappa) \leq \kappa \leq \alpha^{<\kappa}.$$

Hence $\alpha^{<\kappa} = \kappa$, and for each $\nu < \kappa$ we have

$$2^\nu \leq \alpha^\nu < \alpha^{<\kappa} = \kappa,$$

as required. □

Remark 5.19. It is not difficult to show, as in [3, 3.12], that for every uncountable regular cardinal α there is a product space X_I such that $S(X_I) = S((X_I)_\omega) = \alpha$; indeed, as noted there, with $Y := \Pi_{\beta<\alpha} D(\beta)$ one has $S(Y^I) = \alpha$ for all nonempty sets I. Thus the instance $S(X_I) = \alpha$ allowed by Theorem 5.1 does in fact arise in non-trivial circumstances, provided that uncountable regular limit cardinals α do exist. In any case it is immediate from Theorem 5.1 that for every infinite cardinal α of the form $\alpha = \beta^+$ one has $S((D(\beta))^I) = \alpha$ for all nonempty sets I. The κ-box analogue of these statements holds for suitable regular cardinals α (see Theorem 5.23(a) and Remark 5.24(b) below), but the full analogue fails consistently (see Remark 5.37).

We continue with results preparatory to the proof of Theorem 5.23.

Theorem 5.20. *Let $\alpha \geq 3$ and $\kappa \geq \omega$ be cardinals, and let $\{X_i : i \in I\}$ be a set of spaces such that $|I|^+ \geq \kappa$ and $\alpha \leq S(X_i)$ for each $i \in I$. Then*

(a) $\alpha^{<\kappa} \leq S((X_I)_\kappa)$; *and*

(b) *if in addition $\alpha < \kappa$ then $(2^{<\kappa})^+ = (\alpha^{<\kappa})^+ \leq S((X_I)_\kappa)$.*

Proof. (a) We consider two cases.

Case 1. α is singular. Then for each $i \in I$ we have $S(X_i) \geq \alpha^+$ and it follows from Lemma 5.12 (with α^+ now replacing α) that for each $\lambda < \kappa$ we have $S((X_I)_\kappa) > \alpha^\lambda$. Thus

$$S((X_I)_\kappa) \geq \sup_{\lambda<\kappa} (\alpha^\lambda)^+ \geq \Sigma_{\lambda<\kappa}\, \alpha^\lambda = \alpha^{<\kappa}.$$

Case 2. α is regular. (We consider here only the case $\alpha \geq \kappa$ since the case $\alpha < \kappa$ is considered in (b).) Fix $\beta < \alpha$. Since $S(X_i) > \beta$ for each $i \in I$, it follows from Lemma 5.12 that $S((X_I)_\kappa) \geq (\beta^\lambda)^+$ for every $\lambda < \kappa$. Therefore

$$S((X_I)_\kappa) \geq \sup_{\beta<\alpha} (\beta^\lambda)^+ \geq \Sigma_{\beta<\alpha}\, \beta^\lambda. \tag{5.2}$$

Since α is regular and $\lambda < \kappa \leq \alpha$, for each $A \in [\alpha]^\lambda$ there is $\xi < \alpha$ such that $A \subseteq \xi$ (with $|\xi| < \alpha$), so $\alpha^\lambda = \Sigma_{\beta<\alpha}\,\beta^\lambda$. It follows from (5.2) that $S((X_I)_\kappa) \geq \alpha^\lambda$ for each $\lambda < \kappa$. Hence $S((X_I)_\kappa) \geq \alpha^{<\kappa}$, as required.

(b) Since

$$2^{<\kappa} \leq \alpha^{<\kappa} \leq (2^\alpha)^{<\kappa} = \Sigma_{\lambda<\kappa}\,(2^\alpha)^\lambda = \Sigma_{\lambda<\kappa}\,2^\lambda = 2^{<\kappa},$$

we have

$$2^{<\kappa} = \alpha^{<\kappa}.$$

Now, fix $\lambda < \kappa$. Since $S(X_i) > 2$ for each $i \in I$, it follows from Lemma 5.12 that

$$S((X_I)_\kappa) \geq S((X_I)_{\lambda^+}) \geq (2^\lambda)^+. \tag{5.3}$$

Case 1. There exists $\nu < \kappa$ such that $\alpha^\nu = \alpha^{<\kappa}$. Since $\alpha < \kappa$, without loss of generality, we can assume that $\nu \geq \alpha$. Then $\alpha^\nu = 2^\nu$ and from (5.3) we get

$$S((X_I)_\kappa) \geq (2^\nu)^+ > 2^\nu = \alpha^{<\kappa}.$$

Case 2. Case 1 fails. If $\alpha^{<\kappa}$ is regular then it follows from Lemma 5.18(b) that $\kappa = \alpha^{<\kappa}$ and κ is a regular strong limit cardinal. If $\kappa = \omega$ then surely $S((X_I)_\kappa) \geq \kappa^+$, and if $\kappa > \omega$ then Theorem 5.13(a) applies to give

$$S((X_I)_\kappa) \geq \kappa^+ = (\alpha^{<\kappa})^+.$$

Now let $\alpha^{<\kappa}$ be singular. Since (5.3) holds for every $\lambda < \kappa$ we have

$$S((X_I)_\kappa) \geq \sup_{\lambda<\kappa}\,(2^\lambda)^+ \geq \Sigma_{\lambda<\kappa}\,2^\lambda = \alpha^{<\kappa}$$

and since $\alpha^{<\kappa}$ is singular we have $S((X_I)_\kappa) \geq (\alpha^{<\kappa})^+$, as required. □

Corollary 5.21. *Let α, β and κ be cardinals with $\alpha \geq 3$ and $\kappa \geq \omega$, and let $\{X_i : i \in I\}$ be a set of spaces such that $|I|^+ \geq \kappa$, and $d(X_i) \leq \beta$ and $\alpha \leq S(X_i)$ for each $i \in I$. Then $\alpha^{<\kappa} \leq S((X_I)_\kappa) \leq (\beta^{<\kappa})^+$.*

Proof. Follows directly from Theorem 5.14 and Theorem 5.20. □

Corollary 5.22. *Let $\alpha \geq 3$ and $\kappa \geq \omega$ be cardinals, and let $\{X_i : i \in I\}$ be a set of spaces such that $|I|^+ \geq \kappa$ and $d(X_i) \leq \alpha \leq S(X_i)$ for each $i \in I$. Then $\alpha^{<\kappa} \leq S((X_I)_\kappa) \leq (\alpha^{<\kappa})^+$.*

Corollary 5.22 provides tight parameters, but leaves undetermined the question exactly when it is that the value of $S((X_I)_\kappa)$ is $\alpha^{<\kappa}$ and when it is $(\alpha^{<\kappa})^+$. In the following theorem we settle that matter completely.

Theorem 5.23. *Let $\alpha \geq 3$ and $\kappa \geq \omega$ be cardinals, and let $\{X_i : i \in I\}$ be a set of spaces such that $|I|^+ \geq \kappa$ and $d(X_i) \leq \alpha \leq S(X_i)$ for each $i \in I$. Consider these conditions:* (i) *α is regular;* (ii) *$\alpha = \alpha^{<\kappa}$;* (iii) *$\kappa \ll \alpha$;* (iv) *$S((X_J)_\kappa) = \alpha$ for all nonempty $J \in [I]^{<\kappa}$. Then:*

(a) *if conditions* (i), (ii), (iii) *and* (iv) *hold, then $S((X_I)_\kappa) = \alpha^{<\kappa} = \alpha$; and*

(b) *if one (or more) of conditions* (i), (ii), (iii) *or* (iv) *fails, then $S((X_I)_\kappa) = (\alpha^{<\kappa})^+$.*

Proof. (a) is immediate from Theorem 5.7, since $S(X_i) = \alpha = \alpha^{<\kappa}$ for each $i \in I$ under the present hypotheses.

(b) It suffices, according to Corollary 5.22, to assume that $S((X_I)_\kappa) = \alpha^{<\kappa}$ and to show that conditions (i), (ii), (iii) and (iv) must hold. We consider two cases.

Case 1. There is $\nu < \kappa$ such that $\alpha^\nu = \alpha^{<\kappa}$. We fix such ν.

If (i) fails then $S(X_i) = \alpha^+$ and from Lemma 5.12 (with α^+ and ν in the roles of α and μ, respectively) we have $S((X_I)_\kappa) \geq S((X_I)_{\nu^+}) > \alpha^\nu = \alpha^{<\kappa}$, a contradiction. Thus (i) holds.

To see that (ii) holds, suppose first that there are $\beta < \alpha$ and $\lambda < \kappa$ such that $\beta^\lambda \geq \alpha$. Then

$$\alpha^{<\kappa} = \alpha^\nu \leq \beta^{\lambda\cdot\nu} \leq \alpha^{\lambda\cdot\nu} \leq \alpha^{<\kappa}$$

so from Lemma 5.12 we conclude that $S((X_I)_\kappa) > \beta^{<\kappa} \geq \beta^{\lambda\cdot\nu} = \alpha^{<\kappa}$, a contradiction. Thus

$$\beta^\lambda < \alpha \text{ for all } \beta < \alpha, \lambda < \kappa. \tag{5.4}$$

It follows that $\kappa \leq \alpha$. Then each $\lambda < \kappa$ satisfies $\lambda < \alpha$ and from the regularity of α we have $[\alpha]^\lambda = \bigcup_{\beta<\alpha} [\beta]^\lambda$ for each such λ. Thus (5.4) gives

$$\alpha^{<\kappa} = \Sigma_{\lambda<\kappa}\, \alpha^\lambda \leq \Sigma_{\lambda<\kappa}[\Sigma_{\beta<\alpha}\, \beta^\lambda] \leq \kappa \cdot \alpha \cdot \alpha = \alpha \leq \alpha^{<\kappa},$$

and (ii) is proved. To show (iii) we need only show $\kappa < \alpha$, since (5.4) then gives $\kappa \ll \alpha$. Suppose then that $\kappa = \alpha$. Then (5.4) shows that κ is a (regular, strong) limit cardinal, so from Theorem 5.13 we have the contradiction $S((X_I)_\kappa) > \kappa = \alpha = \alpha^{<\kappa}$. Thus $\kappa < \alpha$ and the proof of (iii) is complete. To prove (iv), it suffices to note that if $S((X_J)_\kappa) > \alpha$ for some nonempty $J \in [I]^{<\kappa}$, then we have the contradiction $S((X_I)_\kappa) > \alpha = \alpha^{<\kappa}$.

Case 2. There is no $\nu < \kappa$ such that $\alpha^\nu = \alpha^{<\kappa}$. If $\alpha^{<\kappa}$ is singular then $S((X_I)_\kappa) = \alpha^{<\kappa}$ is impossible, so $\alpha^{<\kappa}$ is regular and Lemma 5.18(b) applies to show that $\alpha^{<\kappa} = \kappa$ is a (regular, strong) limit cardinal; from Theorem 5.13 we again have the contradiction $S((X_I)_\kappa) > \kappa = \alpha^{<\kappa}$. □

Although every infinite Souslin number is regular and uncountable, hence is either a successor cardinal or an uncountable regular limit cardinal, it is perhaps not clear from Theorem 5.23 exactly which uncountable regular cardinals occur in the form $S((X_I)_\kappa)$ with $\{X_i : i \in I\}$ constrained as in Theorem 5.23. Is part (a) of that theorem potentially vacuous? Can every successor cardinal β^+ occur as $\beta^+ = \alpha$ in Theorem 5.23(a)? For each κ, can some $\beta^+ = \alpha$ so occur? Do there exist, for every regular limit cardinal α, infinite $\kappa \ll \alpha$ and spaces $\{X_i : i \in I\}$ such that $S((X_I)_\kappa) = \alpha$? We address these questions in 5.24—5.26 below.

Remarks 5.24. (a) Let β be a singular cardinal and set $\alpha := \beta^+$. Let I be an uncountable set and for $i \in I$ set $X_i := D(\beta)$. Clearly (i), (ii) and (iii) are satisfied with $\kappa = \omega$; also (iv) is satisfied with $\kappa = \omega$, since if $J \in [I]^{<\omega}$ then $X_J = X^J$ is discrete with $|X_J| = \beta$, so $S(X_I) = S((X_I)_\omega) = \beta^+ = \alpha$. Thus $S(X_I) = S((X_I)_\omega) = \alpha$ by Theorem 5.23(a). The same conclusion is available from Theorem 5.23(b) by replacing α everywhere in the statement of Theorem 5.23 by β. In this case both (i) and (iv) fail for β, so $S(X_I) = S((X_I)_\omega) = \beta^+ = \alpha$ by Theorem 5.23(b).

(b) Similar examples exist in ZFC for every uncountable regular cardinal κ. Indeed, given such κ let $\gamma \geq 2$ be arbitrary and set $\beta := \beth_\kappa(\gamma)$ and $\alpha := \beta^+$. For $\lambda < \kappa$ we have

$$[\beta]^\lambda = \bigcup_{\delta<\beta} [\delta]^\lambda \text{ and } [\alpha]^\lambda = \bigcup_{\xi<\alpha} [\xi]^\lambda,$$

so

$$\beta^\lambda \leq \Sigma_{\delta<\beta}\, \delta^\lambda \leq \Sigma_{\delta<\beta}\, 2^\delta \cdot 2^\lambda = \beta$$

and

$$\alpha^\lambda \leq \alpha \cdot \beta^\lambda \leq \alpha \cdot \beta = \alpha.$$

Conditions (i), (ii) and (iii) are then clear and again, as in (a), if $|I| > \kappa$ and $X_i := D(\beta)$ for each $i \in I$, then each space $(X^J)_\kappa = (X_J)_\kappa$ is discrete (when $|J| < \kappa$) with $|X_J| = \beta$, so $S((X_J)_\kappa) = \beta^+ = \alpha$ and (iv) holds by Theorem 5.23(a). Also as in (a) above the same conclusion is available from Theorem 5.23(b) by replacing α everywhere in the statement of Theorem 5.23 by β. In this case both (i) and (iv) fail for β, so $S((X_I)_\kappa) = \beta^+ = \alpha$ by Theorem 5.23(b).

(c) (a) and (b) above indicate that in all models of ZFC conditions (i), (ii), (iii) and (iv) of Theorem 5.23(a) are satisfied by suitably chosen cardinals and spaces, so part (a) of Theorem 5.23 is not vacuous. Those examples depend, however, on choosing for α a regular cardinal of the form

$\alpha = \beta^+$. Part (a) of Theorem 5.25 shows exactly which successor cardinals γ^+ arise as $S((X_I)_\kappa)$ in Theorem 5.23, and part (b) indicates when it can occur that $S((X_I)_\kappa)$ is a limit cardinal.

Theorem 5.25. *Let $\alpha \geq 3$ and $\kappa \geq \omega$ be cardinals, and let $\{X_i : i \in I\}$ be a set of spaces such that $|I|^+ \geq \kappa$ and $d(X_i) \leq \alpha \leq S(X_i)$ for each $i \in I$.*

(a) *If $S((X_I)_\kappa)$ is a successor cardinal—say $S((X_I)_\kappa) = \gamma^+$—then either conditions* (i), (ii), (iii) *and* (iv) *of Theorem 5.23 hold and $\gamma = \gamma^{<\kappa}$, or at least one of those conditions fails and $\gamma = \alpha^{<\kappa}$.*

(b) *If $S((X_I)_\kappa)$ is a (regular) limit cardinal, then $S((X_I)_\kappa) = \alpha$ and $d(X_i) = S(X_i) = \alpha$ for each $i \in I$.*

Proof. (a) If $\gamma^+ = S((X_I)_\kappa) \neq (\alpha^{<\kappa})^+$ then by Theorem 5.23 the indicated conditions (i), (ii), (iii) and (iv) all hold and $\gamma^+ = S((X_I)_\kappa) = \alpha = \alpha^{<\kappa}$. Since $\kappa \ll \alpha = \gamma^+$ we have $\gamma^\lambda = \gamma$ for all $\lambda < \kappa$ and hence $\gamma^{<\kappa} \leq \kappa \cdot \gamma = \gamma$, so $\gamma = \gamma^{<\kappa}$ as asserted.

(b) If $S((X_I)_\kappa)$ is a regular limit cardinal then conditions (i), (ii), (iii) and (iv) of Theorem 5.23 all hold, so $S((X_I)_\kappa) = \alpha$ by Theorem 5.23(a); further for each $i \in I$ we have $S(X_i) = \alpha$ by condition (iv). If there is $i \in I$ such that $d(X_i) < \alpha$ then $(d(X_I))^+ < \alpha = S(X_i)$, which is impossible. □

Remark 5.26. It is well known that the existence of an uncountable regular strong limit cardinal cannot be established in ZFC [19, 12.12], but it should be noted that in case such a cardinal α exists then there are cardinals κ and spaces X_i to which Theorem 5.23(a) and Theorem 5.25 apply. Indeed, let α be a regular limit cardinal and suppose that κ satisfies $\omega \leq \kappa \ll \alpha = \alpha^{<\kappa}$. (These latter conditions are satisfied by every infinite $\kappa < \alpha$, in case α is in addition assumed to be a strong limit cardinal.) Let I be a nonempty set and for $\beta < \alpha$ and $i \in I$ set $D(\beta, i) := D(\beta)$. Then $\{D(\beta, i) : \beta < \alpha, i \in I\}$ is a set of spaces with each $d(D(\beta, i)) = \beta < \alpha$, so by Theorem 5.10 the space $Y := \Pi_{\beta<\alpha, i\in I} D(\beta, i)$ satisfies $S((Y)_\kappa) = \alpha$. As a set we have $Y = X^I$ with $X := \Pi_{\beta<\alpha} D(\beta)$, and the topology of the space Y_κ is finer than the topology of the space $(X^I)_\kappa$, so also the power space X^I satisfies $S((X^I)_\kappa) = \alpha$, where α is a (regular strong) limit cardinal. Clearly $d(X) = \alpha$ and $S(X) = \alpha$ and therefore $(X^I)_\kappa$ is an example of a product space that satisfies all the hypotheses and the conclusion of Theorem 5.23(a).

Lemma 5.27. *Let κ be a strong limit cardinal.*

(a) *If $2 \leq \alpha < \kappa$ then $\alpha^{<\kappa} = \kappa$.*

(b) *If κ is regular then $\kappa^{<\kappa} = \kappa$.*

(c) *If κ is singular then $\kappa^{<\kappa} = 2^\kappa$.*

Proof. (a) We have

$$\kappa \leq \alpha^{<\kappa} \leq (2^\alpha)^{<\kappa} = \Sigma_{\lambda<\kappa}\,(2^\alpha)^\lambda = \Sigma_{\lambda<\kappa}\,2^\lambda \leq \kappa \cdot \kappa = \kappa.$$

(b) This is proved in Lemma 4.18(a).

(c) With $\{\kappa_\eta : \eta < \mathrm{cf}(\kappa)\}$ chosen as in Notation 5.16 we have

$$2^\kappa = 2^{\Sigma_{\eta<\mathrm{cf}(\kappa)}\,\kappa_\eta} = \Pi_{\eta<\mathrm{cf}(\kappa)}\,2^{\kappa_\eta} \leq \kappa^{cf(\kappa)} \leq \kappa^{<\kappa} \leq 2^\kappa.$$

□

Theorem 5.28. *Let κ be a strong limit cardinal and I be an index set with $|I| \geq \kappa$.*

(a) *If $2 \leq \alpha < \kappa$ then $S(((D(\alpha))^I)_\kappa) = \kappa^+$.*

(b) *If κ is regular then $S(((D(\kappa))^I)_\kappa) = \kappa^+$.*

(c) *If κ is singular then $S(((D(\kappa))^I)_\kappa) = (2^\kappa)^+$.*

Proof. In each case, condition (iii) of Theorem 5.23 fails, so parts (a), (b) and (c) follow from Theorem 5.23(b) and from parts (a), (b) and (c) of Lemma 5.27, respectively. □

Remark 5.29. As we noted in Discussion 4.14(f) the value of $d((\mathbf{2}^{\aleph_\omega})_{\aleph_\omega}$ depends on the model of ZFC, while the findings we have enunciated here are sufficiently powerful that the weight and Souslin number of such spaces as $(\mathbf{2}^{\aleph_\omega})_{\aleph_1}$ and $(\mathbf{2}^{\aleph_\omega})_{\aleph_\omega}$ in $\mathbb{V}_1$ and $\mathbb{V}_2$ now emerge painlessly. To make those computations, recall that $2^{\aleph_n} = \aleph_{n+1}$ and $2^{\aleph_\omega} = \aleph_\omega^\omega = \aleph_{\omega+2}$ there, so we have from Theorem 3.7(b)

$$w((\mathbf{2}^{\aleph_\omega})_{\aleph_1}) = \aleph_\omega^\omega = \aleph_{\omega+2},$$

also

$$w((\mathbf{2}^{\aleph_\omega})_{\aleph_\omega}) = (\aleph_\omega)^{<\aleph_\omega} = \aleph_{\omega+2}$$

in both those models.

Concerning the Souslin number, it is clear that $S((\mathbf{2}^{\aleph_0})_{\aleph_1}) = (2^{\aleph_0})^+ = \mathfrak{c}^+$ in ZFC, so from Corollary 5.11 we have for each nonempty set I that

$$\mathfrak{c}^+ = S((\mathbf{2}^{\aleph_0})_{\aleph_1}) \leq S((\mathbf{2}^I)_{\aleph_1}) \leq \mathfrak{c}^+,$$

hence $S((\mathbf{2}^{\aleph_\omega})_{\aleph_1}) = \mathfrak{c}^+$ in ZFC (with $\mathfrak{c}^+ = \aleph_2$ in the models $\mathbb{V}_1$ and $\mathbb{V}_2$).

Finally, from Theorem 5.28(a) (or from Theorem 5.23(b)) we have

$$S((\mathbf{2}^{\aleph_\omega})_{\aleph_\omega}) = (\aleph_\omega)^+ = \aleph_{\omega+1}$$

in $\mathbb{V}_1$ and $\mathbb{V}_2$.

In Theorem 5.33 below we give an upper bound for the Souslin number of a product space with the κ-box topology that depends only on the Souslin numbers of its coordinate spaces, rather than (as in Theorem 5.8) on the Souslin numbers of its "small" sub-products. For that we need the following notation (see [3], [4], [21]).

Notation 5.30. Let α, β, κ and λ be cardinals. The *arrow notation* $\alpha \to (\kappa)^\beta_\lambda$ denotes the following partition relation: if $[\alpha]^\beta = \cup_{i<\lambda} P_i$ then there are $A \subseteq \alpha$ and $i < \lambda$ such that $|A| = \kappa$ and $[A]^\beta \subseteq P_i$.

Preliminary to Theorem 5.33 we give a combinatorial lemma which makes plain the relevance of the arrow relation $\alpha \to (\kappa)^2_\lambda$ to numbers of the form $S((X_I)_{\lambda^+})$. The (general) proof we give is as anticipated in [3, page 73]; it parallels in all its essentials that of the special case treated in [3, 3.13].

We remark that results significantly stronger than that of Lemma 5.31, which have perhaps not received the attention or the recognition they deserve, were developed by Negrepontis and his school in Athens in the 1970's. It is shown in [4, 5.17] for example, using the hypothesis $\omega \leq \lambda < \kappa \ll \alpha$ with κ and α regular, that if each $S(X_i) \leq \kappa$ then not only is $S((X_I)_{\lambda^+}) \leq \alpha$, as in Lemma 5.31, but in fact of every α-many nonempty open subsets of $(X_I)_{\lambda^+}$ some α-many have the finite intersection property.

Lemma 5.31. *Let α, κ and λ be infinite cardinals such that $\alpha \to (\kappa)^2_\lambda$, and let $\{X_i : i \in I\}$ be a set of spaces such that $S(X_i) \leq \kappa$ for each $i \in I$. Then $S((X_I)_{\lambda^+}) \leq \alpha$.*

Proof. Suppose that there is a faithfully indexed cellular family $\{U^\xi : \xi < \alpha\}$ of basic open subsets of $(X_I)_{\lambda^+}$, and for $\{\xi, \xi'\} \in [\alpha]^2$ let $i(\xi, \xi') \in I$ be such that $U^\xi_{i(\xi,\xi')} \cap U^{\xi'}_{i(\xi,\xi')} = \emptyset$. For $\xi < \alpha$ we define

$$I(\xi) := \{i(\xi, \xi') : \xi' < \alpha \text{ and } \xi \neq \xi'\}.$$

Since $i(\xi, \xi') \in R(U^\xi) \cap R(U^{\xi'})$ for $\{\xi, \xi'\} \in [\alpha]^2$ we have $|I(\xi)| \leq |R(U^\xi)| \leq \lambda$ for $\xi < \alpha$. Let $\{i_{\xi,\eta} : \eta < \lambda\}$ be an indexing of $I(\xi)$ for $\xi < \alpha$, and for $(\eta, \eta') \in \lambda \times \lambda$ set

$$P_{\eta,\eta'} := \{\{\xi, \xi'\} \in [\alpha]^2 : \xi < \xi' \text{ and } i_{\xi,\eta} = i_{\xi',\eta'}\}$$

(some of the sets $P_{\eta,\eta'}$ might be empty). Since

$$[\alpha]^2 = \bigcup_{(\eta,\eta') \in \lambda \times \lambda} P_{\eta,\eta'}$$

and $\alpha \to (\kappa)^2_\lambda$, there are $A \in [\alpha]^\kappa$ and $(\overline{\eta}, \overline{\eta'}) \in \lambda \times \lambda$ such that $[A]^2 \subseteq P_{\overline{\eta},\overline{\eta'}}$. Thus there is $\overline{i} \in I$ such that if $\{\xi, \xi'\} \in [A]^2$ and $\xi < \xi'$ then $i(\xi, \xi') = i_{\xi,\overline{\eta}} = i_{\xi',\overline{\eta'}} = \overline{i}$ and hence $U^\xi_{\overline{i}} \cap U^{\xi'}_{\overline{i}} = \emptyset$. It follows that $\{U^\xi_{\overline{i}} : \xi \in A\}$ is cellular in $X_{\overline{i}}$ and we have the contradiction $S(X_{\overline{i}}) > |A| = \kappa$. □

The following theorem is [4, Theorem 1.5(a)]. It is noted in [4] that preliminary formulations of Theorem 5.32 appear (with different hypotheses) in Erdős and Rado [10, 39(iii)] and Kurepa [23]. The important and motivating special case $(2^\alpha)^+ \to (\alpha^+)^2_\alpha$ of Theorem 5.32 appeared as early as 1942 [9], while the seminal instance $\kappa = \omega$, $\alpha = \omega^+$ of Theorem 5.33(a) was given by Kurepa [24] (see also [3, Theorem 3.13 and remark on pp. 73-74]).

Theorem 5.32. *If $\omega \le \kappa \ll \alpha$ with α and κ regular, then $\alpha \to (\kappa)^2_\lambda$ for all $\lambda < \kappa$.*

Theorem 5.33. *Let $\alpha \ge 2$ and $\kappa \ge \omega$ be cardinals and let $\{X_i : i \in I\}$ be a family of nonempty spaces such that $S(X_i) \le \alpha^+$ for each $i \in I$.*

(a) *If $\alpha^+ \ge \kappa$ then $S((X_I)_\kappa) \le (2^\alpha)^+$; and*

(b) *if $\alpha^+ \le \kappa$ then $S((X_I)_\kappa) \le ((\alpha^{<\kappa})^{<\kappa})^+$.*

Proof. (a) Clearly $\alpha \ge \omega$ here, so Remark 5.6(2) applies to give $\alpha^+ \ll (2^\alpha)^+$. We then have

$$(2^\alpha)^+ \to (\alpha^+)^2_\alpha$$

by Theorem 5.32, so $S((X_I)_{\alpha^+}) \le (2^\alpha)^+$ by Lemma 5.31 (with $(2^\alpha)^+$, α^+, and α in the role of α, κ, and λ there). Thus surely $S((X_I)_\kappa) \le (2^\alpha)^+$ if $\kappa \le \alpha^+$.

(b) We consider three cases.

Case 1. κ is singular. Then $\kappa^+ \ll ((\alpha^{<\kappa})^{<\kappa})^+$ by Remark 5.6(4), hence we have from Theorem 5.32 that

$$((\alpha^{<\kappa})^{<\kappa})^+ \to (\kappa^+)^2_\kappa.$$

Since $\alpha \le \kappa$ we have $S(X_i) \le \kappa^+$ for each $i \in I$, so in fact even

$$S((X_I)_\kappa) \le S((X_I)_{\kappa^+}) \le ((\alpha^{<\kappa})^{<\kappa})^+$$

by Lemma 5.31 (with $((\alpha^{<\kappa})^{<\kappa})^+$, κ^+, and κ in the role of α, κ, and λ there).

Case 2. κ is a successor cardinal, say $\kappa = \lambda^+$. Since $\lambda^+ \ll (\alpha^\lambda)^+$, by Theorem 5.32 we have

$$(\alpha^\lambda)^+ \to (\lambda^+)^2_\lambda,$$

so

$$S((X_I)_\kappa) \le (\alpha^\lambda)^+ = ((\alpha^{<\kappa})^{<\kappa})^+$$

by Lemma 5.31 (with $(\alpha^\lambda)^+$, λ^+, and λ in the role of α, κ, and λ there).

Case 3. κ is regular limit cardinal. Then it follows from Remark 5.6(3) that $\omega \leq \kappa \ll ((\alpha^{<\kappa})^{<\kappa})^+$ and therefore, according to Theorem 5.32, $((\alpha^{<\kappa})^{<\kappa})^+ \to (\kappa)^2_\lambda$ for all $\lambda < \kappa$. Then since $S(X_i) \leq \kappa$ for each $i \in I$ we have

$$S((X_I)_{\lambda^+}) \leq ((\alpha^{<\kappa})^{<\kappa})^+ \text{ for all } \lambda < \kappa \tag{5.5}$$

from Lemma 5.31 (with $((\alpha^{<\kappa})^{<\kappa})^+$ in the role of α there).

Now suppose that $\mathcal{C}$ is a cellular family in $(X_I)_\kappa$ of canonical open sets such that $|\mathcal{C}| = ((\alpha^{<\kappa})^{<\kappa})^+$, and for $\lambda < \kappa$ let $\mathcal{C}(\lambda) := \{U \in \mathcal{C} : |R(U)| < \lambda\}$. Then $\mathcal{C} = \bigcup_{\lambda<\kappa} \mathcal{C}(\lambda)$ with

$$\mathrm{cf}(((\alpha^{<\kappa})^{<\kappa})^+) = ((\alpha^{<\kappa})^{<\kappa})^+ \geq \kappa^+ > \kappa,$$

so there is $\lambda < \kappa$ such that $|\mathcal{C}(\lambda)| = ((\alpha^{<\kappa})^{<\kappa})^+$. Then

$$S((X_I)_{\lambda^+}) > |\mathcal{C}(\lambda)| = ((\alpha^{<\kappa})^{<\kappa})^+,$$

contrary to (5.5). □

Remark 5.34. We note that in those cases of Theorem 5.33 to which both (a) and (b) apply, namely when $\kappa = \alpha^+$, the upper bounds provided by the estimates in (a) and (b) coincide. Indeed with $\kappa = \alpha^+$ we have

$$((\alpha^{<\kappa})^{<\kappa})^+ = ((\alpha^\alpha)^\alpha)^+ = (2^\alpha)^+.$$

Combining Theorem 5.20 and Theorem 5.33 we obtain the following corollary.

Corollary 5.35. *Let $\alpha \geq 3$ and $\kappa \geq \omega$ be cardinals, and let $\{X_i : i \in I\}$ be a set of spaces such that $|I|^+ \geq \kappa$ and $\alpha \leq S(X_i) \leq \alpha^+$ for each $i \in I$.*

(a) *If $\alpha^+ \geq \kappa$ then $\alpha^{<\kappa} \leq S((X_I)_\kappa) \leq (2^\alpha)^+$; and*

(b) *if $\alpha^+ \leq \kappa$ then $(2^{<\kappa})^+ \leq S((X_I)_\kappa) \leq ((2^{<\kappa})^{<\kappa})^+$.*

Corollary 5.36. *Let $\alpha \geq 3$ and $\kappa \geq \omega$ be cardinals, and let $\{X_i : i \in I\}$ be a set of spaces such that $|I|^+ \geq \kappa$ and $3 \leq S(X_i) \leq \alpha^+$ for each $i \in I$. If $\alpha^+ \leq \kappa$ then $S((X_I)_\kappa) = (2^{<\kappa})^+$.*

Proof. The case $3 = \alpha < \kappa$ of Theorem 5.20(b) gives $S((X_I)_\kappa \geq (2^{<\kappa})^+$.

If κ is regular or there is $\nu < \kappa$ such that $2^\nu = 2^{<\kappa}$ then $2^{<\kappa} = (2^{<\kappa})^{<\kappa}$ by Theorem 2.8(a) and the statement is immediate from Corollary 5.35(b).

Now we assume that κ is singular and that $2^\nu < 2^{<\kappa}$ for each $\nu < \kappa$ and we let $\{\kappa_\eta : \eta < \mathrm{cf}(\kappa)\}$ be a set of cardinals as in Notation 5.16. Suppose that $S((X_I)_\kappa) > (2^{<\kappa})^+$, and let $\mathcal{C}$ be a cellular family of basic open sets in $(X_I)_\kappa$ with $|\mathcal{C}| = (2^{<\kappa})^+$. Then with

$$\mathcal{C}(\eta) := \{C : C \in \mathcal{C} \text{ and } C \text{ is open in } (X_I)_{\kappa_\eta}\}$$

for $\eta < \mathrm{cf}(\kappa)$ we have $\mathcal{C} = \cup_{\eta<\mathrm{cf}(\kappa)}\mathcal{C}(\eta)$, and since $\mathrm{cf}((2^{<\kappa})^+) > \mathrm{cf}(\kappa)$ there is $\eta < \mathrm{cf}(\kappa)$ such that $|\mathcal{C}(\eta)| = (2^{<\kappa})^+$; for this η we have

$$S((X_I)_{\kappa_\eta}) > (2^{<\kappa})^+. \tag{5.6}$$

Since κ is singular we have from $\alpha^+ \leq \kappa$ that $\alpha < \kappa$, so there is $\eta' < \mathrm{cf}(\kappa)$ such that $\alpha < \kappa_{\eta'}$; we take $\eta' \geq \eta$. Then from Theorem 5.33(b) with $\kappa_{\eta'}^+$ replacing κ we have

$$S((X_I)_{\kappa_\eta}) \leq S((X_I)_{\kappa_{\eta'}^+}) \leq (\alpha^{\kappa_{\eta'}})^+ \leq ((2^\alpha)^{\kappa_{\eta'}})^+ = (2^{\kappa_{\eta'}})^+ < 2^{<\kappa},$$

which contradicts (5.6). □

Remarks 5.37. (a) We noted in Remark 5.24 that the conditions given in Theorem 5.23 are satisfied by many pairs κ, α of cardinals and for many sets $\{X_i : i \in I\}$ of spaces; in particular (see condition (iv) of Theorem 5.23) for $\kappa \ll \alpha = \alpha^{<\kappa}$ there are spaces X such that $S((X^I)_\kappa) = \alpha$ for all nonempty index sets I. We note now that consistently there are (regular) α and κ for which the relation $S((X^I)_\kappa) = \alpha$ holds for no space X and infinite index set I. Indeed, let $\mathbb{V}$ be one of the Gitik-Shelah models whose salient cardinality properties are given in Discussion 4.14(d) and let $\kappa = \aleph_1$ and $\alpha = \aleph_{\omega+1}$. Suppose there is a space X such that $S(X) = \alpha$ and $S((X^I)_\kappa) = \alpha$ for some infinite set I. Then $S(X) = \alpha = \aleph_{\omega+1} > \aleph_\omega$, and it follows from Lemma 5.12 that $S((X^I)_\kappa) > (\aleph_\omega)^\omega = \aleph_{\omega+2}$, a contradiction.

(b) The upper bound $S(X \times X) = (2^\alpha)^+$ for spaces X such that $S(X) = \alpha^+$, allowed by Theorem 5.33(a), is in fact achieved for many α and X. This was first shown by Galvin and Laver (cf. [15]) assuming $\alpha^+ = 2^\alpha$ (see [4, 7.13] for a treatment of the construction) and by examples in ZFC by Todorčević [28], [29], [30]. When $\alpha^+ = 2^\alpha$ this strict increase from $S(X)$ to $S(X \times X)$ is minimal in the sense that

$$S(X \times X) = (2^\alpha)^+ = (\alpha^+)^+ = (S(X))^+.$$

We note in contrast that in the models discussed in (a) there are spaces X such that $S(X^I) \geq (S(X))^{++}$, for every infinite set I. For example, for any space X in those models satisfying $S(X) = \alpha = \aleph_{\omega+1}$ and with $\kappa = \aleph_1$ we have

$$((\aleph_{\omega+1})^+)^+ = \aleph_{\omega+3} \leq S((X^I)_\kappa) \leq (2^\alpha)^+ = (2^{\aleph_{\omega+1}})^+$$

in these models. Similarly Fleissner [14, Section 5], in suitably defined Cohen models of ZFC, constructs spaces X for which $S(X) = \omega^+ = \aleph_1$ and $S(X \times X) = \aleph_{\omega+2} > \mathfrak{c} = \aleph_{\omega+1}$.

The rest of this section is devoted to seeking definitive relations between and among the cardinals $S((X_I)_\kappa)$, $S((X_J)_\kappa)$ with $J \in [I]^{<\kappa}$, and $(\alpha^{<\kappa})^+$. Our success, though substantial, is only partial, since we have been unable to give a fully satisfactory answer to Question 5.45 in ZFC.

Theorem 5.38. *Let $\alpha \geq 2$ and $\kappa \geq \omega$ be cardinals such that $\alpha^{<\kappa} < (\alpha^{<\kappa})^{<\kappa}$. If $\{X_i : i \in I\}$ is a set of nonempty spaces such that $S((X_I)_\kappa) > (\alpha^{<\kappa})^+$, then there are a cardinal $\lambda < \kappa$ and $J \in [I]^{<\lambda}$ such that $S((X_J)_\lambda) \geq (\alpha^{<\kappa})^+$.*

Proof. Let $\mathcal{C}$ be a cellular family of basic open subsets of $(X_I)_\kappa$ such that $|\mathcal{C}| = (\alpha^{<\kappa})^+$. Let $\{\kappa_\eta : \eta < \mathrm{cf}(\kappa)\}$ be as in Notation 5.16 and for $\eta < \mathrm{cf}(\kappa)$ set $\mathcal{C}(\eta) := \{U \in \mathcal{C} : |R(U)| < \kappa_\eta\}$. Since $|\mathcal{C}| = \bigcup_{\eta<\mathrm{cf}(\kappa)} \mathcal{C}(\eta)$ and $\mathrm{cf}((\alpha^{<\kappa})^+) = (\alpha^{<\kappa})^+ \geq \kappa^+ > \mathrm{cf}(\kappa)$, there is $\eta < \mathrm{cf}(\kappa)$ (henceforth fixed) such that $|\mathcal{C}(\eta)| = (\alpha^{<\kappa})^+$. Then $\mathcal{C}(\eta)$ is cellular in $(X_I)_{\kappa_\eta}$, hence in $(X_I)_{\kappa_\eta^+}$, and for $\eta < \eta' < \mathrm{cf}(\kappa)$ we have

$$S((X_I)_{\kappa_\eta^+}) > |\mathcal{C}(\eta)| = (\alpha^{<\kappa})^+ > (\alpha^{\kappa_{\eta'}})^+.$$

Then since $\kappa_\eta^+ \ll (\alpha^{\kappa_{\eta'}})^+$ there is, by Theorem 5.7 (with κ_η^+ and $(\alpha^{\kappa_{\eta'}})^+$ in the roles of κ and α respectively), a set $J(\eta') \in [I]^{<\kappa_\eta^+}$ such that $S((X_{J(\eta')})_{\kappa_\eta^+}) > (\alpha^{\kappa_{\eta'}})^+$. Then with $J := \bigcup_{\eta<\eta'<\mathrm{cf}(\kappa)} J(\eta')$ we have $|J| \leq \kappa_\eta \cdot \mathrm{cf}(\kappa) < \kappa$, and

$$S((X_J)_{\kappa_\eta^+}) > (\alpha^{\kappa_{\eta'}})^+ \text{ when } \eta < \eta' < \mathrm{cf}(\kappa),$$

hence

$$S((X_J)_{\kappa_\eta^+}) \geq \sup_{\eta<\eta'<\mathrm{cf}(\kappa)} (\alpha^{\kappa_{\eta'}})^+ = \Sigma_{\eta<\eta'<\mathrm{cf}(\kappa)} (\alpha^{\kappa_{\eta'}})^+ = \alpha^{<\kappa}.$$

Since in our case $\alpha^{<\kappa}$ is singular (Theorem 2.8(b)) we have

$$S((X_J)_{\kappa_\eta^+}) \geq (\alpha^{<\kappa})^+,$$

so the conclusion holds with $\lambda := \max\{\kappa_\eta^+, |J|^+\}$. □

We continue in Corollary 5.40 with a consequence of Theorem 5.38 for which Lemma 5.39 is preparatory.

Lemma 5.39. *Let $\kappa \geq \omega$ be a limit cardinal, $\alpha \geq 2$ be a cardinal and $\{X_i : i \in I\}$ be a set of nonempty spaces. Then $S((X_J)_\lambda) < \alpha$ for each $\lambda < \kappa$ and each nonempty $J \in [I]^{<\lambda}$ if and only if $S((X_J)_\kappa) < \alpha$ for each nonempty $J \in [I]^{<\kappa}$.*

Proof. Let $S((X_J)_\kappa) < \alpha$ for each nonempty $J \in [I]^{<\kappa}$ and let $\lambda < \kappa$ and $\emptyset \neq J_0 \in [I]^{<\lambda}$. Then $S((X_{J_0})_\lambda) = S((X_{J_0})_\kappa) < \alpha$ since the spaces $(X_{J_0})_\lambda$ and $(X_{J_0})_\kappa$ have the full box topology and therefore coincide.

For the converse, let $S((X_J)_\lambda) < \alpha$ for each $\lambda < \kappa$ and each nonempty $J \in [I]^{<\lambda}$ and let $\emptyset \neq J_0 \in [I]^{<\kappa}$. Since $|J_0| < \kappa$ and κ is a limit cardinal, there exists $\lambda < \kappa$ such that $|J_0| < \lambda$. Then $S((X_{J_0})_\kappa) = S((X_{J_0})_\lambda) < \alpha$ since the spaces $(X_{J_0})_\kappa$ and $(X_{J_0})_\lambda$ have the full box topology and therefore coincide. $\square$

Corollary 5.40. *Let $\alpha \geq 2$ and $\kappa \geq \omega$ be cardinals such that $\alpha^{<\kappa} < (\alpha^{<\kappa})^{<\kappa}$ and let $\{X_i : i \in I\}$ be a set of nonempty spaces such that $S((X_J)_\kappa) < (\alpha^{<\kappa})^+$ for each nonempty $J \in [I]^{<\kappa}$. Then $S((X_I)_\kappa) \leq (\alpha^{<\kappa})^+$.*

Proof. The cardinal κ is singular by Theorem 2.8(b), hence is a limit cardinal. Then by Lemma 5.39 (with α there replaced by $(\alpha^{<\kappa})^+$) we have $S((X_J)_\lambda) < (\alpha^{<\kappa})^+$ whenever $\lambda < \kappa$ and $J \in [I]^{<\lambda}$. Then $S((X_I)_\kappa) \leq (\alpha^{<\kappa})^+$ by Theorem 5.38. $\square$

Theorem 5.42, like Corollary 5.43, is a miscellaneous stand-alone result based on the homeomorphisms developed in Lemma 5.41. To see that those results are (consistently) nonvacuous, we need a model of ZFC where $\alpha^{<\kappa} < (\alpha^{<\kappa})^{<\kappa}$ and $\alpha^\kappa > (\alpha^{<\kappa})^+$. For that, see Remark 5.44. In 5.41–5.43, given a set $\{X_i : i \in I\}$ of spaces, for $i \in I$ we write

$$\widetilde{i} := \{j \in I : X_i =_h X_j\}.$$

We note that if $\kappa \geq \omega$ and $|\widetilde{i}| \geq \mathrm{cf}(\kappa)$ for each $i \in I$, then there is a partition $\{I(\eta) : \eta < \mathrm{cf}(\kappa)\}$ of I such that $|I(\eta) \cap \widetilde{i}| = |\widetilde{i}|$ for each $i \in I$. Indeed, it is enough for each $i \in I$ to choose a partition $\{A(\widetilde{i}, \eta) : \eta < \mathrm{cf}(\kappa)\}$ of $\widetilde{i}$ with each $|A(\widetilde{i}, \eta)| = |\widetilde{i}|$ and to take $I(\eta) := \bigcup_{i \in I} A(\widetilde{i}, \eta)$.

Lemma 5.41. *Let $\kappa \geq \omega$ and $\mathrm{cf}(\kappa) \leq \lambda \leq \kappa$ with λ regular, and let $\{X_i : i \in I\}$ be a set of spaces with each $|\widetilde{i}| \geq \mathrm{cf}(\kappa)$. Let $\{I(\eta) : \eta < \mathrm{cf}(\kappa)\}$ be a partition of I such that $|I(\eta) \cap \widetilde{i}| = |\widetilde{i}|$ for each $i \in I$. Then*

(a) $(X_I)_\lambda =_h (X_{I(\eta)})_\lambda$ *for each* $\eta < \mathrm{cf}(\kappa)$*;*

(b) $(X_I)_\lambda =_h (\Pi_{\eta < \mathrm{cf}(\kappa)} (X_{I(\eta)})_\lambda)_\lambda$*; and*

(c) $(X_I)_\lambda =_h (((X_I)_\lambda)^{\mathrm{cf}(\kappa)})_\lambda$.

Proof. (a) Given $\eta < \mathrm{cf}(\kappa)$, let $\phi : I \twoheadrightarrow I(\eta)$ be a bijection such that $\phi[\widetilde{i}] = \widetilde{i} \cap I(\eta)$ for each $i \in I$. Then the map $\Phi : X_I \twoheadrightarrow X_{I(\eta)}$ given by $\Phi(x_i) = x_{\phi(i)} \in X_{I(\eta)}$ is a homeomorphism, with $\phi[R(A)] = R(\Phi[A])$ for each generalized rectangle $A = \Pi_{i \in I} A_i \subseteq X_I$.

(b) We show that the natural map from $\Pi_{\eta<\mathrm{cf}(\kappa)} X_{I(\eta)}$ onto X_I is a homeomorphism from $(\Pi_{\eta<\mathrm{cf}(\kappa)} (X_{I(\eta)})_\lambda)_\lambda$ onto $(X_I)_\lambda$ when λ is regular. Indeed, an (open) generalized rectangle $U = \Pi_{\eta<\mathrm{cf}(\kappa)} U(\eta)$ in $\Pi_{\eta<\mathrm{cf}(\kappa)} X_{I(\eta)}$ with $U(\eta) = \Pi_{i\in I(\eta)} U(\eta, i)$ satisfies $R(U) = \bigcup_{\eta<\mathrm{cf}(\kappa)} R(U(\eta))$, so $|R(U)| < \lambda$ if and only if each $|R(U(\eta))| < \lambda$.

(c) follows immediately from (a) and (b). □

Theorem 5.42. *Let $\alpha \geq 2$ and $\kappa \geq \omega$ be cardinals and let $\{X_i : i \in I\}$ be a set of nonempty spaces. Suppose that $\alpha^{<\kappa} < (\alpha^{<\kappa})^{<\kappa}$ and that $|\widetilde{i}| \geq \mathrm{cf}(\kappa)$ for each $i \in I$. If $S((X_I)_\kappa) > (\alpha^{<\kappa})^+$ then $S((X_I)_\kappa) > \alpha^\kappa$.*

Proof. By Theorem 5.38 there are $J \in [I]^{<\kappa}$ and a regular cardinal $\lambda < \kappa$ such that $S((X_J)_\lambda) \geq (\alpha^{<\kappa})^+$. Let $\{I(\eta) : \eta < \mathrm{cf}(\kappa)\}$ be a partition of I as in Lemma 5.41, and for $\eta < \mathrm{cf}(\kappa)$ let $\mathcal{C}(\eta)$ be a cellular family in $X_{I(\eta)}$ such that $|\mathcal{C}(\eta)| = \alpha^{<\kappa}$. Then

$$\mathcal{C} := \Pi_{\eta<\mathrm{cf}(\kappa)} \mathcal{C}(\eta) = \{\Pi_{\eta<\mathrm{cf}(\kappa)} C(\eta) : C(\eta) \in \mathcal{C}(\eta)\}$$

is cellular in $(\Pi_{\eta<\mathrm{cf}(\kappa)} (X_{I(\eta)})_\lambda)_\lambda =_h (X_I)_\lambda$, so

$$S((X_I)_\kappa) \geq S((X_I)_\lambda) > |\mathcal{C}| = (\alpha^{<\kappa})^{\mathrm{cf}(\kappa)} = \alpha^\kappa.$$

□

Corollary 5.43. *Let $\alpha \geq 2$ and $\kappa \geq \omega$ be cardinals, and let X be a space and I a set.*

(a) *Suppose that $\alpha^{<\kappa} = (\alpha^{<\kappa})^{<\kappa}$. Then $S((X^I)_\kappa) \leq (\alpha^{<\kappa})^+$ if and only if $S((X^J)_\kappa) \leq (\alpha^{<\kappa})^+$ for every nonempty $J \in [I]^{<\kappa}$.*

(b) *Suppose that $\alpha^{<\kappa} < (\alpha^{<\kappa})^{<\kappa}$. If $S((X^I)_\kappa) > (\alpha^{<\kappa})^+$ then*

(1) *there is nonempty $J \in [I]^{<\kappa}$ such that $S((X^J)_\kappa) \geq (\alpha^{<\kappa})^+$; and*

(2) *if $|I| \geq \mathrm{cf}(\kappa)$, then $S((X^I)_\kappa) > \alpha^\kappa$.*

Proof. In view of Theorem 5.8 and Theorem 5.38, only (b)(2) requires attention. This follows from Theorem 5.42, since now $\widetilde{i} = I$ for each $i \in I$. □

Remarks 5.44. (a) It is easy to see that in many models of ZFC, for example under GCH, the equality $\alpha^\kappa = (\alpha^{<\kappa})^+$ holds for all cardinals α and κ for which $\alpha^{<\kappa} < (\alpha^{<\kappa})^{<\kappa}$. In such models, Theorem 5.42 and Corollary 5.43(b)(2) become tautologies. To see that Theorem 5.42 and Corollary 5.43(b)(2) are not vacuous in every setting, it is enough to refer to the models $\mathbb{V}_1$ and $\mathbb{V}_2$ of Gitik and Shelah described in Discussion 4.14(d), taking now $\alpha = 2$ and $\kappa = \aleph_\omega$. In those models we have

$$\alpha^{<\kappa} = 2^{<\aleph_\omega} = \aleph_\omega,$$

while (using Theorem 2.6(c), for example)

$$\alpha^\kappa = (\alpha^{<\kappa})^{<\kappa} = 2^{\aleph_\omega} = \aleph_{\omega+2} > \aleph_{\omega+1} = (\alpha^{<\kappa})^+.$$

(b) It is a consequence of Theorem 5.42 and Corollary 5.43(b)(2) that under the hypotheses there the relation $S((X_I)_\kappa) = \alpha^\kappa$ is impossible (even when α^κ is regular).

Now we consider two questions. The first of these arises naturally from Corollary 5.8(b) and Corollary 5.40, and a version of the second, attributed to Argyros and Negrepontis, appears in [4]. Theorem 5.48 shows a relation between these. For what we do and do not not know about the status of these questions in ZFC and in augmented systems, see Remarks 5.50((a) and (b)).

Question 5.45. Let $\alpha \geq 2$, $\kappa \geq \omega$, and let $\{X_i : i \in I\}$ be a set of spaces such that $S((X_J)_\kappa) \leq (\alpha^{<\kappa})^+$ for each nonempty $J \in [I]^{<\kappa}$. Is then necessarily $S((X_I)_\kappa) \leq (\alpha^{<\kappa})^+$?

Question 5.46 ([4, 7.15(a)])**.** Are there spaces X and Y with $S(X \times Y) > S(X) > S(Y)$?

Remark 5.47. By Theorem 5.8(b), the answer to Question 5.45 is affirmative in case $\alpha^{<\kappa} = (\alpha^{<\kappa})^{<\kappa}$.

Theorem 5.48. *Let $\alpha \geq 2$, $\kappa \geq \omega$, and $\{X_i : i \in I\}$ witness a negative answer to Question 5.45. If $\alpha < \kappa$ then the answer to Question 5.46 is positive.*

Proof. Set

$$L := \{i \in I : S(X_i) = (\alpha^{<\kappa})^+\} \text{ and } M := \{i \in I : S(X_i) < (\alpha^{<\kappa})^+\}.$$

Note first from Remark 5.47 that $\alpha^{<\kappa} < (\alpha^{<\kappa})^{<\kappa}$, so $\kappa > \omega$, and κ and $\alpha^{<\kappa}$ are singular by Theorem 2.8. Further, it follows directly from our hypothesis that $|I| \geq \kappa$. Clearly we may assume without loss of generality that $S(X_i) \geq 3$ for every $i \in I$.

For each infinite cardinal λ we have $S((X_I)_{\lambda^+}) = S((X_L)_{\lambda^+} \times (X_M)_{\lambda^+})$. Thus to prove the theorem it suffices to show that there exists an infinite cardinal $\lambda < \kappa$ such that

(i) $S((X_L)_{\lambda^+}) = (\alpha^{<\kappa})^+$,

(ii) $S((X_M)_{\lambda^+}) < \alpha^{<\kappa}$, and

(iii) $S((X_I)_{\lambda^+}) > (\alpha^{<\kappa})^+$.

Let $\{\kappa_\eta : \eta < \mathrm{cf}(\kappa)\}$ be a family of cardinals as in Notation 5.16.

We note that

$$|L| < \mathrm{cf}(\kappa), \tag{5.7}$$

and

$$\text{there is } \eta < \mathrm{cf}(\kappa) \text{ such that } S(X_i) \le \alpha^{\kappa_\eta} \text{ for each } i \in M. \tag{5.8}$$

(Indeed if (5.7) [resp., (5.8)] fails then by Theorem 5.17(b) there is $J \in [L]^{\mathrm{cf}(\kappa)}$ [resp., $J \in [M]^{\mathrm{cf}(\kappa)}$] such that

$$S((X_J)_\kappa) \ge S((X_J)_{(\mathrm{cf}(\kappa))^+}) > (\alpha^{<\kappa})^+,$$

a contradiction since $|J| = \mathrm{cf}(\kappa) < \kappa$.)

It follows from (5.7) that $|M| = |I| \ge \kappa$; further, according to Lemma 5.12 (with M, γ^+, 2 and $(\alpha^{<\kappa})^+$ in place of I, κ, β and α, respectively), we have

$$S((X_M)_{\gamma^+}) > 2^\gamma \text{ for every infinite } \gamma < \kappa. \tag{5.9}$$

We claim that

$$S((X_L)_\kappa) = S((X_M)_\kappa) = (\alpha^{<\kappa})^+. \tag{5.10}$$

To see that, fix η as in (5.8) and let γ be such that $\kappa_\eta < \gamma < \kappa$. Since $\alpha < \kappa$, we have $(\alpha^\gamma)^+ < \alpha^{<\kappa}$; then

$$S((X_M)_{\gamma^+}) \le (((\alpha^{\kappa_\eta})^\gamma)^\gamma)^+ = (\alpha^\gamma)^+ < \alpha^{<\kappa} \tag{5.11}$$

by (5.8) and Theorem 5.33(b) (with α, κ and I replaced by α^{κ_η}, γ^+ and M, respectively). It then follows from (5.9) and (5.11) that

$$\sup\{S((X_M)_\gamma) : \gamma < \kappa\} = \alpha^{<\kappa},$$

hence from Theorem 5.13(c) we have

$$S((X_M)_\kappa) = (\alpha^{<\kappa})^+.$$

Since $S((X_M)_\kappa) = (\alpha^{<\kappa})^+ < S((X_I)_\kappa)$ we have $M \ne I$, so $L \ne \emptyset$. Clearly then $S((X_L)_\kappa) \ge (\alpha^{<\kappa})^+$, while $S((X_L)_\kappa) \le (\alpha^{<\kappa})^+$ follows from (5.7) and our hypothesis. Therefore $S((X_L)_\kappa) = (\alpha^{<\kappa})^+$ and claim (5.10) is proved.

From (5.10) we have

$$(\alpha^{<\kappa})^+ \leq S((X_L)_{\lambda^+}) \leq S((X_L)_\kappa) = (\alpha^{<\kappa})^+$$

for each infinite λ, so (i) holds (for all infinite λ). That (ii) holds for all λ such that $\kappa_\eta < \lambda < \kappa$ is given by (5.11). It follows that there is λ such that $\kappa_\eta < \lambda < \kappa$ and $S((X_I)_{\lambda^+}) > (\alpha^{<\kappa})^+$, since otherwise we have

$$\sup\{S((X_I)_{\lambda^+}) : \lambda < \kappa\} = \sup\{S((X_I)_\lambda) : \lambda < \kappa\} = (\alpha^{<\kappa})^+$$

and Theorem 5.13(b) gives the contradiction $S((X_I)_\kappa) = (\alpha^{<\kappa})^+$. Then (iii) holds for that specific λ. □

Corollary 5.49. *Let $\mathbb{M}$ be a model of ZFC in which every singular cardinal is a strong limit cardinal (e.g. $\mathbb{M}$ is a model of ZFC+GCH). If the answer to Question 5.45 is negative in $\mathbb{M}$ then the answer to Question 5.46 is positive in $\mathbb{M}$.*

Proof. With α, κ and $\{X_i : i \in I\}$ chosen as in Theorem 5.48 it suffices to show that $\alpha < \kappa$.

Since $\alpha^{<\kappa} < (\alpha^{<\kappa})^{<\kappa}$ by Remark 5.47, by Theorem 2.8(b), both κ and $\alpha^{<\kappa}$ are singular. If $\alpha = \alpha^{<\kappa}$ then $\alpha = \alpha^{<\kappa} = (\alpha^{<\kappa})^{<\kappa}$, a contradiction; therefore $\alpha < \alpha^{<\kappa}$. Suppose now that $\kappa \leq \alpha$. Then from Theorem 2.6(c) we have $(\alpha^{<\kappa})^{<\kappa} = \alpha^\kappa \leq \alpha^\alpha = 2^\alpha$, and the relation $\alpha < \alpha^{<\kappa} < 2^\alpha$ contradicts the hypothesis that $\alpha^{<\kappa}$ is a strong limit cardinal. □

Remarks 5.50. (a) The proof of the previous corollary does not need the full hypothesis that every singular cardinal in $\mathbb{M}$ is strong limit. It is enough to know just that $\alpha^{<\kappa}$ is strong limit.

(b) ZFC-consistent examples of spaces as requested in Question 5.46 are available in the literature.

(1) In the Cohen models of Fleissner [14] (see Remark 5.37(b)) there are spaces X and Y such that

$$S(Y \times Y) = \aleph_{\omega+2} > \mathfrak{c} = \aleph_{\omega+1} > \aleph_1 = S(Y),$$

and then with X the "disjoint union" of $D(\aleph_1)$ and Y we have $S(X \times Y) > S(X) > S(Y)$.

(2) It is shown by Shelah [27, 4.4] that if κ is a singular strong limit cardinal such that $\lambda := \kappa^+ = 2^\kappa$ then there are spaces X and Y such that

$$S(X \times Y) \geq \lambda^{++} > \lambda^+ = S(X) > \lambda > \kappa > (2^{\mathrm{cf}(\kappa)})^{++} \geq S(Y).$$

(c) We do not know if there are models of ZFC in which no spaces as in Question 5.46 exist. We do not know if the answer to Question 5.45 is absolutely or consistently "Yes", absolutely or consistently "No".

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